

Somekawa's K -groups and additive higher Chow groups

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Abstract

We introduce the Milnor type K -group attached to some algebraic groups including Witt groups over a perfect field as an extension of Somekawa's K -group. We give a description of this K -group associated to the additive group and the multiplicative group by the space of the absolute Kähler differentials, and relate also our Somekawa K -group for the additive group and the Jacobian variety of a curve with a complex determined by the residue maps and the trace maps. The same arguments work for the Mackey product of the additive higher Chow group and the higher Chow group for a scheme. This gives vanishing results of additive Chow groups on zero-cycles.

1 Introduction

S. Bloch and H. Esnault in [6] introduced the additive higher Chow groups $\mathrm{TCH}^p(k, q) = \mathrm{TCH}^p(k, q; 1)$ for a field k . as an attempt at a cycle-theoretic description of the motivic cohomology theory of a non-reduced scheme. They showed that this additive Chow group is isomorphic to the space of the absolute Kähler differentials of k when $q = p$ as $\mathrm{TCH}^q(k, q) \simeq \Omega_k^{q-1}$ (see Thm. 5.3 for more general results allowing modulus given by K. Rülling [19]). This theorem is an analogue of the theorem of Nesterenko-Suslin/Totaro $\mathrm{CH}^q(k, q) \simeq K_q^M(k)$, where $\mathrm{CH}^q(k, p)$ is the higher Chow group and $K_q^M(k)$ is the Milnor K -group. One of the reasons that the space of the Kähler differentials appeared can be found in the calculation of $K_2(k[t]/(t^2))$ (*cf.* [6],

Sect. 2). The aim of this note is to make this analogy

$$K_q^M(k) \longleftrightarrow \Omega_k^{q-1}$$

clear by extending Somekawa's K -group [23]. More precisely, Let \mathbb{W}_m be the big Witt group of length m and G_1, \dots, G_q split semi-abelian varieties over a perfect field k . Here, a split semi-abelian variety is a product of an abelian variety and a torus. We introduce a group called also the *Somekawa K -group* by

$$K(k; \mathbb{W}_m, G_1, \dots, G_q) := \left(\bigoplus_{E/k: \text{finite}} \mathbb{W}_m(E) \otimes G_1(E) \otimes \dots \otimes G_q(E) \right) / R$$

where R is the subgroup which produces “the projection formula” and “the Weil reciprocity law” as in the Milnor K -theory (Def. 3.3). Under the assumption that the field k is perfect, we realize that the residue map

$$\partial_P : \Omega_{k(C)}^1 \rightarrow k(P)$$

for the function field $k(C)$ of a curve C over k can take its value not in k but in the residue field $k(P)$ at a closed point P of C as the boundary map on the Milnor K -group $\partial_P : K_2(k(C)) \rightarrow k(P)^\times$ does. This fact leads to a condition like the Weil reciprocity law exactly by the same way as in Somekawa's original K -groups. However, the projection formula above implies that our Somekawa K -group becomes trivial when k has positive characteristic and $q \geq 1$ (Lem. 3.5). Thus our main interest here is on the case of characteristic 0. For $\mathbb{G}_a = \mathbb{W}_1$ the additive group and $G_i = \mathbb{G}_m$ the multiplicative groups for $i \geq 1$, this group has the following description:

Theorem 1.1 (Thm. 3.6). *As k -vector spaces, we have*

$$K(k; \mathbb{G}_a, \overbrace{\mathbb{G}_m, \dots, \mathbb{G}_m}^{q-1}) \xrightarrow{\simeq} \Omega_k^{q-1}.$$

This isomorphism should be compared to Somekawa's theorem ([23], Thm. 1.4)

$$K(k; \overbrace{\mathbb{G}_m, \dots, \mathbb{G}_m}^q) \xrightarrow{\simeq} K_q^M(k).$$

Taking \mathbb{G}_a and the Jacobian variety J_X of a projective smooth curve X over k , we relate the K -group $K(k; \mathbb{G}_a, J_X)$ and the complex defined by the residue maps and the trace maps

$$\Omega_{k(X)}^1 \xrightarrow{\partial} \bigoplus_{P \in X_0} k(P) \xrightarrow{\text{Tr}} k,$$

where X_0 is the set of closed points in X , $\partial := \bigoplus_P \partial_P$ and $\text{Tr} := \sum_P \text{Tr}_{k(P)/k}$. The fact that the sequence above becomes a complex is just the so-called residue formula (Thm. 2.6). The Somekawa K group associated to \mathbb{G}_a and the Jacobian of X is written by this complex as follows:

Theorem 1.2 (Thm. 4.5). *We assume that X has a k -rational point. Then there is a canonical isomorphism*

$$K(k; \mathbb{G}_a, J_X) \xrightarrow{\simeq} \text{Ker}(\text{Tr}) / \text{Im}(\partial).$$

From the analogy we are pursuing, the right in this theorem: the homology group $\text{Ker}(\text{Tr}) / \text{Im}(\partial)$ of the above complex is an additive version of the group $V(X)$ defined by Bloch which is used in the proof of the higher dimensional class field theory [4], [20].

In terms of algebraic cycles, the big Witt group \mathbb{W}_m and the multiplicative group \mathbb{G}_m are isomorphic to the additive Chow group $\mathcal{TCH}^1(k, 1; m)$ and the higher Chow group $\mathcal{CH}^1(k, 1)$ of the field k respectively as Mackey functors (we recall the definition in Section 3). Here $\mathcal{CH}^a(X, b)$ and $\mathcal{TCH}^a(X, b; m)$ are the Mackey functors associated to the higher Chow group $\text{CH}^a(X, b)$ and the additive higher Chow group with modulus $\text{TCH}^a(X, b; m)$ for a scheme X over k . (The Mackey functors $\mathcal{CH}^a(\text{Spec } k, b)$ and $\mathcal{TCH}^a(\text{Spec } k, b; m)$ are abbreviated as $\mathcal{CH}^1(k, b)$ and $\mathcal{TCH}^1(k, b; m)$ respectively in the case of $X = \text{Spec } k$.) The isomorphisms $\mathbb{W}_m \simeq \mathcal{TCH}^1(k, 1; m)$ and $\mathbb{G}_m \simeq \mathcal{CH}^1(k, 1)$ as Mackey functors are also compatible with their *local symbols*. Thus we have an isomorphism

$$K(k; \mathbb{W}_m, \mathbb{G}_m, \dots, \mathbb{G}_m) \simeq K(k; \mathcal{TCH}^1(k, 1; m), \mathcal{CH}^1(k, 1), \dots, \mathcal{CH}^1(k, 1))$$

and one can regard the Somekawa K -group as the group associated to the Mackey functors $\mathcal{TCH}^1(k, 1; m)$ and $\mathcal{CH}^1(k, 1)$. In [18], Raskind and Spiess introduced the Milnor K -theory for the higher Chow groups of schemes

$$K(k; \mathcal{CH}^{a_1}(X_1, b_1), \dots, \mathcal{CH}^{a_q}(X_q, b_q))$$

by the similar way to the Somekawa K -group. The key ingredient to define the above group are the projection formula of the intersection product on higher Chow groups and some reciprocity law formulated by the connecting homomorphism of a localization sequence (see [18], Rem. 2.4.2). In order to study the Chow group of 0-cycles on the product of schemes, they showed the following isomorphism:

$$K(k; \mathcal{CH}^{d_1+b_1}(X_1, b_1), \dots, \mathcal{CH}^{d_q+b_q}(X_q, b_q)) \xrightarrow{\cong} \mathrm{CH}^{\delta+\beta}(X_1 \times \dots \times X_q, \beta),$$

where $\beta = \sum_i b_i$ and $\delta = \sum_i d_i$. However, the localization property is not known on the additive Chow groups due to the lack of homotopy invariance. We only consider the Mackey product

$$\mathcal{TCH}^a(X, b; m) \overset{M}{\otimes} \mathcal{CH}^{a_1}(X_1, b_1) \overset{M}{\otimes} \dots \overset{M}{\otimes} \mathcal{CH}^{a_q}(X_q, b_q)(k)$$

which is defined by factoring out a relation concerning the projection formula only. For the moment we only present the following surjectivity on 0-cycles.

Theorem 1.3 (Thm. 5.4). *For projective smooth varieties X and X_i over k of d and d_i dimension respectively, there is a canonical surjection*

$$\begin{aligned} \mathcal{TCH}^{d+b}(X, b; m) \overset{M}{\otimes} \mathcal{CH}^{d_1+b_1}(X_1, b_1) \overset{M}{\otimes} \dots \overset{M}{\otimes} \mathcal{CH}^{d_q+b_q}(X_q, b_q)(k) \\ \twoheadrightarrow \mathrm{TCH}^{\delta+\beta}(X \times X_1 \times \dots \times X_q, \beta; m), \end{aligned}$$

where $\beta = b + \sum_i b_i$ and $\delta := d + \sum_i d_i$.

As byproducts we obtain vanishing results $\mathrm{TCH}^{d+q}(X, q; m) = 0$ for $q \geq 2$ if the base field k is perfect and has positive characteristic. This is an additive version of Akhtar's theorem on the higher Chow groups $\mathrm{CH}^{d+q}(X, q) = 0$ for $q \geq 2$ over a finite field k (cf. Thm. 5.2).

Now we give an outline of this paper. In Section 2, we recall some properties of the generalized Witt group for a field k written as $\mathbb{W}_S(k)$ which is a generalization of both of the ordinary Witt group and the big Witt group following [9], Section 1. We also recall the generalized de Rham-Witt complex $\mathbb{W}_S \Omega_k^1$ following [9], a generalization of Illusie's de Rham-Witt complex [11]. After introducing R ulling's trace maps and residue maps on the generalized de Rham-Witt complex ([19]) we define the residue map (in our sense) $\partial_P : \Omega_{k(C)}^1 \rightarrow k(P)$ as noted above under the assumption that the base field

k is perfect (1). Some functorial properties which we will need in the sequel are also given (Lem. 2.8, Lem. 2.9).

In Section 3, first we recall the definition and some properties of Mackey functors. Combining Somekawa's local symbol and the residue map, we introduce the Somekawa K -group of the form $K(k; \mathbb{W}_S, G_1, \dots, G_q)$ for split semi-abelian varieties G_1, \dots, G_q over a perfect field k as a quotient of the Mackey products $\mathbb{W}_S \overset{M}{\otimes} G_1 \overset{M}{\otimes} \dots \overset{M}{\otimes} G_q(k)$ (Def. 2.10). Finally, we will show that this group, in particular, has the following descriptions:

$$\begin{aligned} K(k; \mathbb{W}_S) &\xrightarrow{\cong} \mathbb{W}_S(k) \quad (\text{Lem. 3.4}), \\ K(k; \mathbb{W}_S, G_1, \dots, G_q) &= 0, \text{ if } \text{char}(k) > 2, q > 1 \quad (\text{Lem. 3.5}), \\ K(k; \mathbb{G}_a, \mathbb{G}_m, \dots, \mathbb{G}_m) &\xrightarrow{\cong} \Omega_k^{q-1} \quad (\text{Thm. 3.6}). \end{aligned}$$

In Section 4, we study the additive counterpart of Section 2 in [23]. First the Somekawa K -groups will be extended following Akhtar [2]. Precisely, we introduce the Milnor type K -group allowing the Chow group of 0-cycles as its coefficients such as $K(k; \mathbb{W}_S, G_1, \dots, G_q, \mathcal{CH}_0(X))$ for a variety X over k (Def. 4.1). Using this group, we will show that

$$\begin{aligned} K(k; \mathbb{W}_S, \mathcal{CH}_0(\text{Spec } k)) &\xrightarrow{\cong} \mathbb{W}_S(k) \quad (\text{Lem. 4.3 (i)}), \\ K(k; \mathbb{W}_S, \mathcal{CH}_0(X)) &\xrightarrow{\cong} \mathbb{W}_S(k), \text{ if } \dim(X) = 1 \text{ and } \text{char}(k) > 2 \quad (\text{Lem. 4.3 (ii)}), \\ K(k; \mathbb{G}_a, \mathcal{CH}_0(X)) &\xrightarrow{\cong} \text{Coker} \left(\partial : \Omega_{k(X)}^1 \rightarrow \bigoplus_{P \in X_0} k(P) \right) \quad (\text{Thm. 4.5}). \end{aligned}$$

Section 5 is devoted to studying the analogy using additive higher Chow groups. We briefly recall the cubical construction of the higher Chow groups and the additive higher Chow groups of schemes following [14] and show the intersection product gives the surjection from the Mackey products to the additive Chow groups on the level of the 0-cycles as stated above (Thm. 5.4).

Recently F. Ivorra and K. Rülling in [12] give the Somekawa-type K -group for more general Mackey functors than ours named the *reciprocity functors* including algebraic groups with unipotent part over a field which is finitely generated over a perfect field. They show also the similar computations with Theorem 1.1. However our proof is more elementary like Somekawa's original proof thus we think that it is indispensable to leave the theorem be in this paper.

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2 Witt groups and local symbol

A subset $S \subset \mathbb{N}$ of the set of positive integers $\mathbb{N} := \mathbb{Z}_{>0}$ is called a *truncation set* if $S \neq \emptyset$ and if for any $n \in S$, all its divisors are in S . The *generalized Witt ring* $\mathbb{W}_S(A)$ is defined to be A^S as a set whose ring structure is determined by the condition that the ghost map

$$\text{gh} : \mathbb{W}_S(A) \rightarrow A^S$$

defined by $(a_s)_{s \in S} \mapsto (w_s)_{s \in S}$ where $w_s := \sum_{d|s} da_d^{s/d}$, is a natural transformation of functors of rings ([9], Sect. 1). We denote by $[-] : A \rightarrow \mathbb{W}_S(A)$ the Teichmüller map defined by $[a] := (a, 0, 0, \dots)$. Here, we give some elementary properties of the Witt rings.

- Lemma 2.1.** (i) *The Teichmüller map is multiplicative, that is, $[ab] = [a][b]$ for any $a, b \in A$. The unit and zero in $\mathbb{W}_S(A)$ are $[1]$ and $[0]$ respectively.*
(ii) *Any ring homomorphism $\phi : A \rightarrow B$ induces the ring homomorphism $\mathbb{W}_S(A) \rightarrow \mathbb{W}_S(B)$ defined by $(a_s)_{s \in S} \mapsto (\phi(a_s))_{s \in S}$. If ϕ is surjective (resp. injective), then the same holds on Witt rings.*
(iii) *For two truncation sets $T \subset S$, the restriction map $\mathbb{W}_S(A) \rightarrow \mathbb{W}_T(A)$ defined by $(a_s)_{s \in S} \mapsto (a_s)_{s \in T}$ is surjective.*

For each $n \in \mathbb{N}$, $S/n := \{d \in \mathbb{N} \mid nd \in S\}$ is a truncation set. The Frobenius and Verschiebung

$$F_n : \mathbb{W}_S(A) \rightarrow \mathbb{W}_{S/n}(A), \quad V_n : \mathbb{W}_{S/n}(A) \rightarrow \mathbb{W}_S(A)$$

are defined by the rules $\text{gh}_s \circ F_n = \text{gh}_{sn}$ and

$$(V_n(a))_s := \begin{cases} a_{s/n}, & \text{if } n \mid s, \\ 0, & \text{otherwise,} \end{cases}$$

respectively. It is known that the following properties hold.

Lemma 2.2. (i) $F_1 = \text{Id}$, $F_m \circ F_n = F_{mn}$ and $V_1 = \text{Id}$, $V_m \circ V_n = V_{mn}$, where Id is the identity map.

(ii) For $w = (w_s)_{s \in S} \in \mathbb{W}_S(A)$,

$$w = \sum_{s \in S} V_s([w_s]).$$

(iii) $(m, n) = 1$, then $F_m \circ V_n = V_n \circ F_m$.

(iv) $F_n \circ V_n = n$.

(v) $F_m \circ V_n([a]) = (m, n)V_{n/(m, n)}([a]^{m/(m, n)})$ for $a \in A$.

(vi) $V_n([a])V_r([b]) = (n, r)V_{nr/(n, r)}([a]^{r/(n, r)}[b]^{n/(n, r)})$ for $a, b \in A$.

(vii) $[a]V_n(w) = V_n([a]^n w)$ for $a \in A$ and $w \in \mathbb{W}_S(A)$.

In particular, we define $\mathbb{W}_m(A) := \mathbb{W}_{\{1, 2, \dots, m\}}(A)$ called the *big Witt ring* of length m . When we fix an odd prime p , $W_m(A) := \mathbb{W}_{\{1, p, \dots, p^{m-1}\}}(A)$ is the ordinary Witt ring (e.g., [21], Chap. II, Sect. 6).

Let p be an odd prime or 0. For a $\mathbb{Z}_{(p)}$ -algebra A and a truncation set S , the *de Rham-Witt complex* $\mathbb{W}_S \Omega_A^\bullet$ is a differential graded algebra over A which generalizes the de Rham complex Ω_A^\bullet and comes equipped with the Verschiebung $V_n : \mathbb{W}_{S/n} \Omega_A^\bullet \rightarrow \mathbb{W}_S \Omega_A^\bullet$ and the Frobenius $F_n : \mathbb{W}_S \Omega_A^\bullet \rightarrow \mathbb{W}_{S/n} \Omega_A^\bullet$ for each $n > 0$. The construction of $\mathbb{W}_S \Omega_A^\bullet$ and some properties are given in [10] and [11]. If S is finite, there is a surjective map of differential graded ring $\Omega_{\mathbb{W}_S(A)/\mathbb{Z}}^\bullet \rightarrow \mathbb{W}_S \Omega_A^\bullet$ and the map becomes bijective when $S = \{1\}$ or in degree 0, namely, $\mathbb{W}_{\{1\}} \Omega_A^\bullet \simeq \Omega_{A/\mathbb{Z}}^\bullet$ is the absolute de Rham complex and $\mathbb{W}_S \Omega_A^0 \simeq \mathbb{W}_S(A)$ is the Witt ring. Now we fix a field k of characteristic p and recall some properties of the de Rham-Witt complex over the field k . The following propositions play important roles in this note.

Proposition 2.3 ([9], Sect. 1.2, [19], Prop. 1.19). (i) Set $P := \{1, p, p^2, \dots\}$ and for a differential graded algebra (Ω, d) and a positive integer j we denote $\Omega(j^{-1}) := (\Omega, j^{-1}d)$. For a truncation set S ,

$$\mathbb{W}_S \Omega_k^\bullet \xrightarrow{\simeq} \prod_{j \in \mathbb{N}, (j, p)=1} \mathbb{W}_{P \cap S/j} \Omega_k^\bullet(j^{-1}).$$

(ii) For a finite separable field extension E/k , there is an isomorphism of $\mathbb{W}_S(E)$ -modules $\mathbb{W}_S(E) \otimes_{\mathbb{W}_S(k)} \mathbb{W}_S \Omega_k^q \simeq \mathbb{W}_S \Omega_E^q$ for all q .

From the isomorphism $\mathbb{W}_S(E) \otimes_{\mathbb{W}_S(k)} \mathbb{W}_S \Omega_k^q \simeq \mathbb{W}_S \Omega_E^q$ the trace map $\text{Tr}_{E/k} : \mathbb{W}_S(E) \rightarrow \mathbb{W}_S(k)$ on the Witt groups is extended to the de Rham-Witt complexes:

Theorem 2.4 ([19], Thm. 2.6). *Let E/k be a finite field extension. Then there is a map of differential graded $\mathbb{W}_S \Omega_k^\bullet$ -modules*

$$\text{Tr}_{E/k} = \text{Tr}_{E/k}^\bullet : \mathbb{W}_S \Omega_E^\bullet \rightarrow \mathbb{W}_S \Omega_k^\bullet$$

satisfying the following properties:

- (a) $\text{Tr}_{E/k}^0 : \mathbb{W}_S(E) \rightarrow \mathbb{W}_S(k)$ is the trace map on Witt rings and the trace map coincides with the one on the de Rham complex $\text{Tr}_{E/k} : \Omega_E^\bullet \rightarrow \Omega_k^\bullet$ ([16]) if $S = \{1\}$.
- (b) If E/k is a separable extension, then we identify $\mathbb{W}_S(E) \otimes \mathbb{W}_S \Omega_k^q \simeq \mathbb{W}_S \Omega_E^q$ and the map $\text{Tr}_{E/k}^q$ is given by $\text{Tr}_{E/k}^0 \otimes \text{Id}$, where Id is the identity map.
- (c) For finite field extensions $k \subset E_1 \subset E_2$, we have $\text{Tr}_{E_2/k} = \text{Tr}_{E_1/k} \circ \text{Tr}_{E_2/E_1}$.
- (d) The trace map commutes with V_n, F_n and restriction maps.

The classical residue map $\Omega_{k((t))}^1 \rightarrow k$ for the field of Laurent series $k((t))$ over a field k ([22], Chap. II, Sect. 11; [16], Sect. 17) is also extended to the de Rham-Witt complexes.

Proposition 2.5 ([19], Prop. 2.12). *Let k be a field and S a finite truncation set. There is a map $\text{Res} = \text{Res}_t^q : \mathbb{W}_S \Omega_{k((t))}^q \rightarrow \mathbb{W}_S \Omega_k^{q-1}$ satisfying the following properties:*

- (a) $\text{Res}_t^q(\alpha\omega) = \alpha \text{Res}_t^{q-i}(\omega)$ for $\alpha \in \mathbb{W}_S \Omega_k^i$ and $\omega \in \mathbb{W}_S \Omega_{k((t))}^{q-i}$.
- (b) Res is a natural transformation with respect to k .
- (c) Res commutes with d, V_n, F_n and restriction.
- (d) If $u \in (k[[t]])^\times$ and $\tau = tu$, then $\text{Res}_t = \text{Res}_\tau$.
- (e) $\text{Res}(\omega) = 0$ for $\omega \in \mathbb{W}_S \Omega_{k[[t]]}^q$ or $\omega \in \mathbb{W}_S \Omega_{k[1/t]}^q$.
- (f) If $\omega \in \mathbb{W}_S \Omega_{k[[t]]}^{q-1}$, then $\text{Res}(\omega \text{dlog}[t]) = \omega(0)$, where $[t] := (t, 0, \dots, 0)$ is the Teichmüller lift of t , $\text{dlog}[t] = d[t]/[t]$ and $\omega(0)$ is the image of ω by the natural map $\mathbb{W}_S \Omega_{k[[t]]}^{q-1} \rightarrow \mathbb{W}_S \Omega_k^{q-1}$.
- (g) For $a, b \in k$ and $i, j \in \mathbb{Z}$,

$$\begin{aligned} \text{Res}(V_n([at^j])dV_m([bt^i])) = \\ \begin{cases} \text{sgn}(i)(i, j)V_{mn/(m,n)}([a]^{m/(m,n)}[b]^{n/(m,n)}), & \text{if } jm + in = 0 \text{ and } i \neq 0 \\ 0, & \text{otherwise,} \end{cases} \end{aligned}$$

where $\text{sgn}(i) = i/|i|$.

From now on, we assume that the field k is *perfect* of characteristic p , where p is an odd prime or 0. For a projective smooth curve C over k , let $K = k(C)$ be the function field of C . For each closed point P of C , let K_P be the fraction field of the completion $\widehat{\mathcal{O}}_{C,P}$ of $\mathcal{O}_{C,P}$ by the normalized valuation v_P and $k(P)$ the residue field of K_P . From the assumption on k , the complete discrete valuation field K_P is canonically isomorphic to $k(P)((t))$. This gives us the residue map

$$(1) \quad \partial_P = \partial_P^q : \mathbb{W}_S \Omega_K^q \rightarrow \mathbb{W}_S \Omega_{K_P}^q \xrightarrow{\text{Res}_t^q} \mathbb{W}_S \Omega_{k(P)}^{q-1}$$

whose target is $\mathbb{W}_S \Omega_{k(P)}^\bullet$ not $\mathbb{W}_S \Omega_k^\bullet$. By the above proposition, ∂_P is independent on the choice of t . The ordinary residue map in [19] (and also [16]) is the composition $\text{Res}_P := \text{Tr}_{k(P)/k} \circ \partial_P$ and it is known the following theorem.

Theorem 2.6 (Trace formula, [19], Thm. 2.19). *Let k be a perfect field and S a finite truncation set. The residue map $\partial_P : \mathbb{W}_S \Omega_{K_P}^q \rightarrow \mathbb{W}_S \Omega_{k(P)}^{q-1}$ satisfies*

$$\sum_{P \in C} \text{Tr}_{k(P)/k} \circ \partial_P(\omega) = 0$$

for any $\omega \in \mathbb{W}_S \Omega_K^q$.

More precisely, the residue map is calculated as follows: The de Rham-Witt complex $\mathbb{W}_S \Omega_{K_P}^q$ decomposes into the p -typical de Rham-Witt complexes $W_m \Omega_{K_P}^q := \mathbb{W}_{\{1,p,\dots,p^{m-1}\}} \Omega_{K_P}^q$ (Prop. 2.3). Hence it is enough to consider the case of the p -typical de Rham-Witt complex $W_m \Omega_{K_P}^q$. Any element $\omega \in W_m \Omega_{K_P}^q$ can be written uniquely as

$$(2) \quad \omega = \sum_{j \in \mathbb{Z}} a_{0,j}[t]^j + b_{0,j}[t]^{j-1}d[t] + \sum_{j \in \mathbb{Z} \setminus p\mathbb{Z}, 1 \leq s < m} V^s(a_{s,j}[t]^j) + dV^s(b_{s,j}[t]^j).$$

for some $a_{i,j} \in W_{m-i} \Omega_{k(P)}^q$, $b_{i,j} \in W_{m-i} \Omega_{k(P)}^{q-1}$ ($a_{i,j} = b_{i,j} = 0$ for $j < 0$) and here $V^s := V_{p^s} : W_{m-s} \Omega_{K_P}^q \rightarrow W_m \Omega_{K_P}^q$ is the Verschiebung ([19], Lem. 2.9, see also Rem. 2.10). In this case, the residue map is given by $\partial_P(\omega) := b_{0,0}$.

Definition 2.7. (i) For Any element $\omega \in W_m \Omega_{K_P}^q$ written as in (2), the *valuation* of ω at P is defined by

$$v_P(\omega) := \begin{cases} \min\{j \mid b_{0,j} \neq 0\} - 1, & \text{if } \omega \neq 0, \\ \infty, & \text{if } \omega = 0. \end{cases}$$

In particular, $\partial_P(\omega) = 0$ if $v_P(\omega) \geq 0$.

(ii) The natural inclusion $K \hookrightarrow K_P$ gives the *local symbol*

$$\partial_P : \mathbb{W}_S(K) \otimes K^\times \rightarrow \mathbb{W}_S(k(P)); w \otimes f \mapsto \partial_P(wd\log[f])$$

where $d\log[f] := d[f]/[f]$. The image of $w \otimes f \in \mathbb{W}_S(K) \otimes K^\times$ by the local symbol map is denoted by $\partial_P(w, f) := \partial_P(wd\log[f])$. This is essentially Witt's residue symbol (*cf.* [19], Rem. 2.13).

Note that the local symbol ∂_P satisfies the conditions of Serre's local symbol ([22], Chap. 3, Sect. 1). In particular, we have

$$(3) \quad \partial_P(w, f) = v_P(f)w(P)$$

if w is in $\mathbb{W}_S(\widehat{\mathcal{O}}_{C,P})$, where $w(P)$ is the image of w by the canonical map $\mathbb{W}_S(\widehat{\mathcal{O}}_{C,P}) \rightarrow \mathbb{W}_S(k(P))$. In terms of the Milnor K -groups, the valuation map v_P is the boundary map $\partial_P : K_1(K) \rightarrow K_0(k(P))$. More generally, the boundary map $\partial_P : K_q^M(K) \rightarrow K_{q-1}^M(k(P))$ on the Milnor K -groups ([3], Prop. 4.3) and the residue map $\partial_P : \mathbb{W}_S\Omega_K^q \rightarrow \mathbb{W}_S\Omega_{K_P}^{q-1}$ are compatible in the following sense: For some field F , we denote the image of $x_1 \otimes \cdots \otimes x_q \in F^\times \otimes \cdots \otimes F^\times$ in $K_q^M(F)$ by $\{x_1, \dots, x_q\}$ as usual. For $q > 0$, define

$$d\log : K_q^M(F) \rightarrow \mathbb{W}_S\Omega_F^q$$

by $\{x_1, \dots, x_q\} \mapsto d\log[x_1] \cdots d\log[x_q]$. When $q = 0$, $d\log : \mathbb{Z} \rightarrow \mathbb{W}_S(F)$ is the canonical map. The equation (3) leads to the following commutative diagram.

Lemma 2.8. *Let C be a projective smooth curve over k and P a closed point of C . Then the following diagram is commutative:*

$$\begin{array}{ccc} K_q^M(k(C)) & \xrightarrow{d\log} & \mathbb{W}_S\Omega_{k(C)}^q \\ \partial_P \downarrow & & \downarrow \partial_P \\ K_{q-1}^M(k(P)) & \xrightarrow{d\log} & \mathbb{W}_S\Omega_{k(P)}^{q-1}. \end{array}$$

Proof. From the equality (3), we may assume $q > 1$. By considering the completion $k(C)_P$ at P , it is enough to show that the following diagram is

commutative

$$\begin{array}{ccc} K_q^M(K) & \xrightarrow{\text{dlog}} & \mathbb{W}_S \Omega_K^q \\ \partial \downarrow & & \downarrow \text{Res}_t^q \\ K_{q-1}^M(k) & \xrightarrow{\text{dlog}} & \mathbb{W}_S \Omega_k^{q-1}. \end{array}$$

for a local field $K = k((t))$. It is known that the Milnor K -group $K_q^M(K)$ is generated by symbols of the form $\{u_1, \dots, u_{q-1}, f\}$ with $v_K(u_i) = 0$ for all i and $f \in K^\times$. For an element $\{u_1, \dots, u_{q-1}, f\}$ of this form, we have $\partial(\{u_1, \dots, u_{q-1}, f\}) = v_K(f)\{\bar{u}_1, \dots, \bar{u}_{q-1}\}$ in $K_{q-1}^M(k)$, where \bar{u}_i is the image of u_i in k^\times ([3], Prop. 4.5). On the other hand, write $f = ut^{v_K(f)}$ with $u \in \mathcal{O}_K^\times$.

$$\begin{aligned} \text{Res}_t \circ \text{dlog}(\{u_1, \dots, u_{q-1}, f\}) &= \text{Res}_t(\text{dlog}[u_1] \cdots \text{dlog}[u_{q-1}] \text{dlog}[u]) \\ &\quad + v_K(f) \text{Res}_t(\text{dlog}[u_1] \cdots \text{dlog}[u_{q-1}] \text{dlog}[t]) \\ &\stackrel{(*)}{=} v_K(f) \text{dlog}[\bar{u}_1] \cdots \text{dlog}[\bar{u}_{q-1}]. \end{aligned}$$

Here, the last equality $(*)$ follows from Proposition 2.5 (e) and (f). \square

Lemma 2.9. *Let $C' \rightarrow C$ be a dominant morphism of smooth projective curves over k . For any closed point P' in C' , we denote by the point P in C defined by the image of P' . Then, we have the following commutative diagram*

$$\begin{array}{ccc} \mathbb{W}_S \Omega_{k(C')}^1 & \xrightarrow{\partial_{P'}} & \mathbb{W}_S(k(P')) \\ \uparrow & & \uparrow e(P'/P) \\ \mathbb{W}_S \Omega_{k(C)}^1 & \xrightarrow{\partial_P} & \mathbb{W}_S(k(P)), \end{array}$$

where the left vertical map is the natural map and the right one is the map multiplication by the ramification index $e(P'/P)$ at P' over P .

Proof. By considering the completions $k(C)_P$ and $k(C')_{P'}$, it is enough to show, for a finite extension K'/K of local fields with residue field extension k'/k ,

$$\begin{array}{ccc} \mathbb{W}_S \Omega_{K'}^1 & \xrightarrow{\partial_{K'}} & \mathbb{W}_S(k') \\ \uparrow & & \uparrow e \\ \mathbb{W}_S \Omega_K^1 & \xrightarrow{\partial_K} & \mathbb{W}_S(k), \end{array}$$

where ∂_K and $\partial_{K'}$ are residue maps and e is the ramification index of K'/K . The de Rham-Witt complex $\mathbb{W}_S\Omega_K^1$ decomposes into the p -typical de Rham-Witt complexes $W_m\Omega_K^1$ and considering the restriction maps it is enough to show the equality

$$(4) \quad \partial_{K'}(\omega) = e \partial_K(\omega)$$

for $\omega \in W_m\Omega_K^1$. First we assume that K'/K is unramified. If we fix a uniformizer t of K , then $K' \simeq k'((t))$ and $K \simeq k((t))$. Since the residue map is natural (Prop. 2.5), we have $\partial_{K'}(\omega) = \partial_K(\omega) \in W_m(k')$ for any $\omega \in W_m\Omega_K^1$. Next we consider the case of the extension K'/K is totally ramified, then let t' be a uniformizer of K' . Now $t = (t')^e$ is a uniformizer of K . From (2), any element $\omega \in W_m\Omega_K^1$ is written of the form

$$\begin{aligned} \omega &= \sum_{j \in \mathbb{Z}} a_{0,j}[t]^j + b_{0,j}[t]^{j-1}d[t] + \sum_{\substack{j \in \mathbb{Z} \setminus p\mathbb{Z}, \\ 1 \leq s < m}} V^s(a_{s,j}[t]^j) + dV^s(b_{s,j}[t]^j) \\ &= \sum_{j \in \mathbb{Z}} a_{0,j}[t']^{ej} + b_{0,j}[t']^{e(j-1)}d[(t')^e] + \sum_{\substack{j \in \mathbb{Z} \setminus p\mathbb{Z}, \\ 1 \leq s < m}} V^s(a_{s,j}[t']^{ej}) + dV^s(b_{s,j}[t']^{ej}) \\ &= \sum_{j \in \mathbb{Z}} a_{0,j}[t']^{ej} + eb_{0,j}[t']^{ej-1}d[t'] + \sum_{\substack{j \in \mathbb{Z} \setminus p\mathbb{Z}, \\ 1 \leq s < m}} V^s(a_{s,j}[t']^{ej}) + dV^s(b_{s,j}[t']^{ej}). \end{aligned}$$

Therefore $\partial_{K'}(\omega) = eb_{0,0} = e \partial_K(\omega)$ (Prop. 2.5 (g)). Finally, for arbitrary finite extension K'/K , let K^{ur} be the maximal unramified subextension of K' over K . Then the required equality (4) follows from

$$\partial_{K'}(\omega) = e \partial_{K^{\text{ur}}}(\omega) = e \partial_K(\omega)$$

for any $\omega \in W_m\Omega_K^1$. □

3 Somekawa K -groups

A *Mackey functor* A is a co- and contravariant functor from the category of étale schemes over a perfect field k to the category of abelian groups such that $A(X_1 \sqcup X_2) = A(X_1) \oplus A(X_2)$ and if

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow f \\ Y' & \xrightarrow{g} & Y \end{array}$$

is a Cartesian diagram, then the induced diagram

$$\begin{array}{ccc} A(X') & \xrightarrow{g'_*} & A(X) \\ f'^* \uparrow & & \uparrow f^* \\ A(Y') & \xrightarrow{g_*} & A(Y) \end{array}$$

commutes. It is uniquely determined by its value $A(E) := A(\text{Spec}(E))$ on finite field extensions over k . A typical example we mainly used here is the one which arises from a G_k -module A defined by $E \mapsto A^{G_E} = H^0(G_E, A)$ for a finite extension field E over k with norm and restriction maps, where $G_k := \text{Gal}(\bar{k}/k)$ is the absolute Galois group of the field k .

Definition 3.1. For Mackey functors A_1, \dots, A_q , their *Mackey product* $A_1 \otimes^M \dots \otimes^M A_q(k)$ is defined by the quotient

$$A_1 \otimes^M \dots \otimes^M A_q(k) := \left(\bigoplus_{E/k: \text{finite}} A_1(E) \otimes \dots \otimes A_q(E) \right) / R,$$

where R is the subgroup generated by elements of the following form:

(PF) For any finite field extensions $k \subset E_1 \subset E_2$, and if $x_{i_0} \in A_{i_0}(E_2)$ and $x_i \in A_i(E_1)$ for $i \neq i_0$, then

$$j^*(x_1) \otimes \dots \otimes x_{i_0} \otimes \dots \otimes j^*(x_q) - x_1 \otimes \dots \otimes j_*(x_{i_0}) \otimes \dots \otimes x_q,$$

where $j : \text{Spec}(E_2) \rightarrow \text{Spec}(E_1)$ is the canonical map.

This product gives a tensor product in the abelian category of the Mackey functors with unit $\mathbb{Z} : E \mapsto \mathbb{Z}$. Note also that the product $- \otimes^M A$ is right exact for any Mackey functor A . We write $\{x_1, \dots, x_q\}_{E/k}$ for the image of $x_1 \otimes \dots \otimes x_q \in A_1(E) \otimes \dots \otimes A_q(E)$ in the product. In particular, for any field extension k'/k and the canonical map $j = j_{k'/k} : k \hookrightarrow k'$, the pull-back

$$\text{Res}_{k'/k} := j^* : A_1 \otimes^M \dots \otimes^M A_q(k) \longrightarrow A_1 \otimes^M \dots \otimes^M A_q(k')$$

is called the *restriction map*. We recall here the construction of the restriction $\text{Res}_{k'/k}$. For any finite extension E/k , we have $E \otimes_k k' = \bigoplus_{i=1}^n A_i$ where A_i

is an Artinian algebra of dimension e_i over the residue field E_i . Denoting $j_i := j_{E_i/E} : E \hookrightarrow E_i$, the restriction map is

$$\text{Res}_{k'/k}(\{x_1, \dots, x_q\}_{E/k}) := \sum_{i=1}^n e_i \{j_i^*(x_1), \dots, j_i^*(x_q)\}_{E_i/k'}.$$

If the extension k'/k is finite, then the push-forward

$$N_{k'/k} := j_* : A_1^{\otimes M} \cdots \otimes A_q^{\otimes M}(k') \longrightarrow A_1^{\otimes M} \cdots \otimes A_q^{\otimes M}(k)$$

is given by $N_{k'/k}(\{x_1, \dots, x_q\}_{E'/k'}) = \{x_1, \dots, x_q\}_{E'/k}$ on symbols and is called the *norm map*.

An algebraic group G over k forms a Mackey functor. For a field extension E_2/E_1 , the pull-back is the canonical map given by $j : E_1 \hookrightarrow E_2$ which is denoted by $j^* = \text{Res}_{E_2/E_1} : G(E_1) \hookrightarrow G(E_2)$. For simplicity, we identify the elements in $G(E_1)$ with its image in $G(E_2)$ and the restriction map Res_{E_2/E_1} will be sometimes omitted. If the extension E_2/E_1 is finite, the push-forward is written as $j_* = N_{E_2/E_1} : G(E_2) \rightarrow G(E_1)$ and is referred to as the norm map. Let G be a semi-abelian variety over k . For the function field $K = k(C)$ of a proper smooth curve C over k , the local symbol map $\partial_P : G(K_P) \otimes K_P^\times \rightarrow G(k(P))$ is defined in [23] by modifying Serre's local symbol map. Denoting by $\partial_P(g, f) := \partial_P(g \otimes f) \in G(k(P))$ it satisfies $\partial_P(g, f) = v_P(f)g(P)$ if $g \in G(\widehat{\mathcal{O}}_{C,P})$, where $g(P)$ is the image of g in $G(k(P))$. As a special case, for the multiplicative group $G = \mathbb{G}_m$, the local symbol map is nothing other than the tame symbol map (= the boundary map $\partial_P : K_2(K) \rightarrow K_1(k(P))$) which is given by

$$(5) \quad \partial_P : \mathbb{G}_m(K) \otimes K^\times \rightarrow \mathbb{G}_m(k(P)); \quad g \otimes f \mapsto (-1)^{v_P(g)v_P(f)} \frac{g^{v_P(f)}}{f^{v_P(g)}}(P).$$

Before the definition of our Somekawa K -groups we introduce one notation.

Definition 3.2. For a projective smooth curve C over k , let $K = k(C)$ be the function field of C . Let $G = T \times A$ be a split semi-abelian variety with torus T and an abelian variety A over k . For any closed point P in C , $f \in K^\times$, and $g = (g', x) \in T(\widehat{\mathcal{O}}_{C,P}) \times A(\widehat{\mathcal{O}}_{C,P}) = G(\widehat{\mathcal{O}}_{C,P})$, let L/K_P be a finite unramified extension with residue field E such that $T_E \simeq (\mathbb{G}_m)^{\oplus n}$. We

denote by $(g_i)_{1 \leq i \leq n} \in \mathbb{G}_m(\mathcal{O}_L)^{\oplus n}$ the image of g' by $T(\widehat{\mathcal{O}}_{C,P}) \hookrightarrow T(\mathcal{O}_L) \simeq \mathbb{G}_m(\mathcal{O}_L)^{\oplus n}$, where \mathcal{O}_L the valuation ring of L . The *valuation*

$$v_P : K^\times \otimes G(\widehat{\mathcal{O}}_{C,P}) \rightarrow \mathbb{Z} \cup \{\infty\}$$

is defined by

$$v_P(f, g) := v_P(f \otimes g) := \min\{v_L(\mathrm{dlog} f \mathrm{dlog} g_i / g_i(P)) \mid 1 \leq i \leq n\},$$

Here $g_i(P) \in \mathbb{G}_m(k(P))$ denotes the image of g_i under the canonical map $\mathbb{G}_m(\widehat{\mathcal{O}}_{C,P}) \rightarrow \mathbb{G}_m(k(P))$.

Definition 3.3. Let S be a finite truncation set and G_1, \dots, G_q split semi-abelian varieties over k . The Milnor type K -group $K(k; \mathbb{W}_S, G_1, \dots, G_q)$ which we also call the *Somekawa K -group* is given by the quotient

$$K(k; \mathbb{W}_S, G_1, \dots, G_q) := \left(\mathbb{W}_S^{\otimes M} \otimes G_1^{\otimes M} \otimes \dots \otimes G_q^{\otimes M}(k) \right) / R$$

modulo the subgroup R generated by elements of the following form:

(WR) Let $K = k(C)$ be the function field of a proper non-singular algebraic curve C over k . Put $G_0 := \mathbb{W}_S$. For $f \in K^\times, g_i \in G_i(K)$, assume that for each closed point P in C there exists $i(P)$ ($0 \leq i(P) \leq q$) such that $g_i \in G_i(\widehat{\mathcal{O}}_{C,P})$ for all $i \neq i(P)$. If $i(P) = 0$, we further assume that $v_P(f, g_i) \geq -v_P(g_0)$ for all $i \neq 0$. Then

$$\sum_{P \in C_0} g_0(P) \otimes \dots \otimes \partial_P(g_{i(P)}, f) \otimes \dots \otimes g_q(P) \in R.$$

Here C_0 is the set of closed points in C .

We also write $\{x, x_1, \dots, x_q\}_{E/k}$ for the image of $x \otimes x_1 \otimes \dots \otimes x_q \in \mathbb{W}_S(E) \otimes G_1(E) \otimes \dots \otimes G_q(E)$ in $K(k; \mathbb{W}_S, G_1, \dots, G_q)$. By multiplication on the first argument, $K(k; \mathbb{W}_S, G_1, \dots, G_q)$ has the structure of $\mathbb{W}_S(k)$ -module. The various functorial properties of the Somekawa K -groups hold as stated in [23]: For an extension k'/k , set $K(k'; \mathbb{W}_S, G_1, \dots, G_q) := K(k'; \mathbb{W}_S \otimes k', G_1 \otimes_k k', \dots, G_q \otimes_k k')$. The restriction map on the Mackey product induces a canonical homomorphism

$$\mathrm{Res}_{k'/k} : K(k; \mathbb{W}_S, G_1, \dots, G_q) \longrightarrow K(k'; \mathbb{W}_S, G_1, \dots, G_q).$$

If the extension k'/k is finite, then the norm map

$$N_{k'/k} : K(k'; \mathbb{W}_S, G_1, \dots, G_q) \longrightarrow K(k; \mathbb{W}_S, G_1, \dots, G_q)$$

is defined on symbols by $N_{k'/k}(\{x, x_1, \dots, x_q\}_{E'/k'}) = \{x, x_1, \dots, x_q\}_{E'/k}$.

From now on, we give some descriptions of the Somekawa K -group attached to some special family of commutative algebraic groups.

Lemma 3.4. *There exists an isomorphism*

$$K(k; \mathbb{W}_S) \xrightarrow{\simeq} \mathbb{W}_S(k).$$

Proof. We define a map $\phi : \mathbb{W}_S(k) \rightarrow K(k; \mathbb{W}_S)$ by $\phi(w) := \{w\}_{k/k}$. This is surjective from the property (PF). For the algebraic closure \bar{k} of k , considering the commutative diagram

$$\begin{array}{ccc} \mathbb{W}_S(k) & \xrightarrow{\phi} & K(k; \mathbb{W}_S) \\ \downarrow & & \downarrow \\ \mathbb{W}_S(\bar{k}) & \xrightarrow{\phi} & K(\bar{k}; \mathbb{W}_S) \end{array}$$

we may assume $k = \bar{k}$. The Somekawa K -group $K(k; \mathbb{W}_S)$ is now the quotient of $\mathbb{W}_S(k)$ by the subgroup generated by the relations of the form (WR). By the trace formula (Thm. 2.6), the local symbol map satisfies $\Sigma_P \partial_P(w, f) = 0$ in $\mathbb{W}_S(k)$ for some function field $K = k(C)$ of one variable over k , $w \in \mathbb{W}_S(K)$ and $f \in K^\times$. The assertion follows from it. \square

It is known that $G_1 \overset{M}{\otimes} \dots \overset{M}{\otimes} G_q(k) = 0$ for semi-abelian varieties G_i and hence $K(k; G_1, \dots, G_q) = 0$ for a finite field k and $q > 1$ ([13]). However, we show the Somekawa K -group becomes trivial for arbitrary perfect field of positive characteristic. This result later gives some vanishing of the additive higher Chow groups of varieties (Cor. 5.6).

Lemma 3.5. *Let k be a perfect field of characteristic $p > 2$. Let G_1, \dots, G_q be semi-abelian varieties over k for $q > 0$, and S a truncation set. Then $\mathbb{W}_S \overset{M}{\otimes} G_1 \overset{M}{\otimes} \dots \overset{M}{\otimes} G_q(k) = 0$ in the Mackey product. In particular, we have*

$$K(k; \mathbb{W}_S, G_1, \dots, G_q) = 0$$

for split semi-abelian varieties G_1, \dots, G_q .

Proof. It is enough to show $\mathbb{W}_S \otimes^M G(k) = 0$ for a semi-abelian variety G over k . Since the generalized Witt group is embedded into a finite product of the Witt groups (Prop. 2.3) and the restriction map from the product of the Witt groups to the generalized Witt group is surjective, the assertion is reduced to showing $W_m \otimes^M G(k) = 0$. Any symbol $\{w, g\}_{E/k}$ in $W_m \otimes^M G(k)$ is in the image of the norm map $N_{E/k} : W_m \otimes^M G(E) \rightarrow W_m \otimes^M G(k)$, without loss of generality, we may assume $E = k$ and show $\{w, g\}_{k/k} = 0$. Let $p^m : G \rightarrow G$ be the isogeny defined by the multiplication by p^m on G . Since the trace maps on the Witt groups are surjective for the finite extension field $E := k(\text{Ker}(p^m : G \rightarrow G))$ over k , there exists $\tilde{w} \in W_m(E)$ such that $\text{Tr}_{E/k}(\tilde{w}) = w$. The projection formula (PF) implies

$$\begin{aligned} \{w, g\}_{k/k} &= \{\text{Tr}_{E/k}(\tilde{w}), g\}_{k/k} \\ &= \{\tilde{w}, j^*(g)\}_{E/k} \\ &= \{\tilde{w}, [p^m]\tilde{g}\}_{E/k} \\ &= \{p^m\tilde{w}, \tilde{g}\}_{E/k} \\ &= 0, \end{aligned}$$

where $j = j_{E/k} : k \hookrightarrow E$ and $\tilde{g} \in G(\bar{k})$ such that $p^m(\tilde{g}) = g$. \square

Recall that the isomorphism $\psi : K(k; \overbrace{\mathbb{G}_m, \dots, \mathbb{G}_m}^q) \xrightarrow{\sim} K_q^M(k)$ is given by $\psi(\{x_1, \dots, x_q\}_{E/k}) = N_{E/k}(\{x_1, \dots, x_q\})$, where $N_{E/k}$ is the norm map on the Milnor K -groups ([23], Thm. 1.4). The inverse map ϕ is $\phi(\{x_1, \dots, x_q\}) = \{x_1, \dots, x_q\}_{k/k}$. In the same manner here we show that the Somekawa K -group taking the additive group $\mathbb{G}_a = \mathbb{W}_1$ and the multiplicative groups \mathbb{G}_m coincides with the space of Kähler differentials.

Theorem 3.6. *Let k be a perfect field of characteristic $\neq 2$. As k -vector spaces, we have a canonical isomorphism*

$$\psi : K(k; \mathbb{G}_a, \overbrace{\mathbb{G}_m, \dots, \mathbb{G}_m}^{q-1}) \xrightarrow{\sim} \Omega_k^{q-1}.$$

As the Milnor K -group $K_2^M(F)$ of a field F is defined by the quotient of $F^\times \otimes F^\times$ modulo the Steinberg relation $x \otimes (1 - x)$ for $x \in F^\times$, the space of the absolute Kähler differential $\Omega_F^1 := \Omega_{F/\mathbb{Z}}^1 = \mathbb{W}_{\{1\}}\Omega_F^1$ has the so called

Cathelineau relation $x \operatorname{dlog} x + (1-x) \operatorname{dlog}(1-x) = 0$. Precisely, a surjective homomorphism of F -vector spaces

$$(6) \quad F \otimes \overbrace{F^\times \otimes \cdots \otimes F^\times}^{q-1} \rightarrow \Omega_F^{q-1}$$

is defined by $x \otimes x_1 \otimes \cdots \otimes x_{q-1} \mapsto x \operatorname{dlog} x_1 \cdots \operatorname{dlog} x_{q-1} = x \operatorname{dlog} x_1 \wedge \cdots \wedge \operatorname{dlog} x_{q-1}$ where $\operatorname{dlog} x := dx/x$ and its kernel is generated by elements of the forms ([5], Lem. 4.1, [7], Lem. 4.2):

$$\begin{aligned} & x \otimes x_1 \otimes \cdots \otimes x_{q-1}, \quad \text{with } x_i = x_j \text{ for some } i < j, \quad \text{or} \\ & x \otimes x \otimes x_2 \otimes \cdots \otimes x_{q-1} + (1-x) \otimes (1-x) \otimes x_2 \otimes \cdots \otimes x_{q-1}. \end{aligned}$$

Proof of Thm. 3.6. From Lemma 3.4, we may assume $q > 1$. Since we are assuming the field k is perfect, the both sides are trivial if k has positive characteristic (Lem. 3.5). Hence we assume that k has characteristic 0.

Definition of ϕ : Define a map

$$\phi : \Omega_k^{q-1} \rightarrow K(k; \mathbb{G}_a, \mathbb{G}_m, \dots, \mathbb{G}_m)$$

by $\phi(x \operatorname{dlog} x_1 \cdots \operatorname{dlog} x_{q-1}) := \{x, x_1, \dots, x_{q-1}\}_{k/k}$. To show that the map ϕ is well-defined, by (6) we need to show

$$(7) \quad \begin{aligned} & \{x, x_1, \dots, x_{q-1}\}_{k/k} = 0 \quad \text{if } x_i = x_j \text{ for some } i < j, \quad \text{and} \\ & \{x, x, x_2, \dots, x_{q-1}\}_{k/k} + \{(1-x), (1-x), x_2, \dots, x_{q-1}\}_{k/k} = 0. \end{aligned}$$

Recall that the structure of a k -vector space on the Somekawa K -group is defined by $a\{x, x_1, \dots, x_{q-1}\}_{E/k} = \{ax, x_1, \dots, x_{q-1}\}_{E/k}$ for $a \in k$. To show the first equality, it is enough to show $\{1, x_1, \dots, x_{q-1}\}_{k/k} = 0$ if $x_i = x_j$ for some $i < j$. The element $\{1, x_1, \dots, x_{q-1}\}_{k/k}$ is in the image of the homomorphism

$$\operatorname{dlog} : K(k; \overbrace{\mathbb{G}_m, \dots, \mathbb{G}_m}^{q-1}) \rightarrow K(k; \mathbb{G}_a, \overbrace{\mathbb{G}_m, \dots, \mathbb{G}_m}^{q-1})$$

defined by $\{y_1, \dots, y_{q-1}\}_{E/k} \mapsto \{1, y_1, \dots, y_{q-1}\}_{E/k}$. From the isomorphism $K(k; \mathbb{G}_m, \dots, \mathbb{G}_m) \xrightarrow{\cong} K_{q-1}^M(k)$, we have $\{1, x_1, \dots, x_{q-1}\}_{k/k} = 0$. For the Cathelineau relation (=the second equality in (7)), we may assume $x \neq 1$. Consider the function field $K = k(\mathbb{P}_k^1) = k(t)$. Take

$$f = t^{-1}, g = t, g_1 = t^{-2}(t-x)(t-(1-x)), \text{ and } g_k = x_k \text{ for } k \geq 2.$$

For any $P \neq P_\infty := \text{infinite place in } K$, $g \in \widehat{\mathcal{O}}_{\mathbb{P}^1, P}$ and we have

$$\begin{aligned} & \{g(P), \partial_P(g_1, f), g_2(P), \dots, g_{q-1}(P)\}_{k/k} \\ &= \begin{cases} \{x, x, x_2, \dots, x_{q-1}\}_{k/k}, & P = (t - x), \\ \{1 - x, 1 - x, x_2, \dots, x_{q-1}\}_{k/k}, & P = (t - (1 - x)), \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

For $P = P_\infty$, we also have $\{\partial_P(g, f), g_1(P), \dots, g_{q-1}(P)\}_{k/k} = 0$. The relation of type (WR) gives the desired result.

Definition of ψ : Now we define a homomorphism

$$\psi : K(k; \mathbb{G}_a, \mathbb{G}_m, \dots, \mathbb{G}_m) \rightarrow \Omega_k^{q-1}$$

by

$$\psi(\{x, x_1, \dots, x_{q-1}\}_{E/k}) := \text{Tr}_{E/k}(x \, \text{dlog } x_1 \cdots \text{dlog } x_{q-1}),$$

where $\text{Tr}_{E/k} : \Omega_E^{q-1} \rightarrow \Omega_k^{q-1}$ is the trace map (Thm. 2.4). To show that ψ is well-defined, first consider the relations of type (PF). For a finite extension E_2/E_1 , and if $x_{i_0} \in \mathbb{G}_m(E_2)$, $x \in \mathbb{G}_a(E_1)$, $x_i \in \mathbb{G}_m(E_1)$ ($i \neq i_0$), by the properties of the trace map (Thm. 2.4), we have

$$\begin{aligned} \psi(\{x, x_1, \dots, x_{q-1}\}_{E_2/k}) &= \text{Tr}_{E_2/k}(x \, \text{dlog } x_1 \cdots \text{dlog } x_{i_0} \cdots \text{dlog } x_{q-1}) \\ &= \text{Tr}_{E_1/k} \circ \text{Tr}_{E_2/E_1}(x \, \text{dlog } x_1 \cdots \text{dlog } x_{i_0} \cdots \text{dlog } x_{q-1}) \\ &\stackrel{(*)}{=} \text{Tr}_{E_1/k}(x \, \text{dlog } x_1 \cdots \text{dlog } N_{E_2/E_1} x_{i_0} \cdots \text{dlog } x_{q-1}) \\ &= \psi(\{x, x_1, \dots, N_{E_2/E_1} x_{i_0}, \dots, x_{q-1}\}_{E_1/k}). \end{aligned}$$

Here, for the third equality $(*)$ above, we used the compatibility of the trace map and the norm map (cf. [16], Sect. 16) which is given by the following commutative diagram:

$$\begin{array}{ccc} E_2^\times & \xrightarrow{\text{dlog}} & \Omega_{E_2}^1 \\ N_{E_2/E_1} \downarrow & & \downarrow \text{Tr}_{E_2/E_1} \\ E_1^\times & \xrightarrow{\text{dlog}} & \Omega_{E_1}^1, \end{array}$$

Similarly, if $x \in \mathbb{G}_a(E_2)$ and $x_i \in \mathbb{G}_m(E_1)$, we have

$$\begin{aligned} \psi(\{x, x_1, \dots, x_{q-1}\}_{E_2/k}) &= \text{Tr}_{E_2/k}(x \, \text{dlog } x_1 \cdots \text{dlog } x_{q-1}) \\ &= \text{Tr}_{E_1/k} \circ \text{Tr}_{E_2/E_1}(x \, \text{dlog } x_1 \cdots \text{dlog } x_{q-1}) \\ &= \text{Tr}_{E_1/k}((\text{Tr}_{E_2/E_1} x) \, \text{dlog } x_1 \cdots \text{dlog } x_{q-1}) \\ &= \psi(\{\text{Tr}_{E_2/E_1} x, x_1, \dots, x_{q-1}\}_{E_1/k}). \end{aligned}$$

Next, for the relation (WR), let $K = k(C)$ be the function field of a curve C over k . Take $f \in K^\times$, $g_0 \in \mathbb{G}_a(K)$, $g_i \in \mathbb{G}_m(K)$ ($1 \leq i \leq q-1$), as in Definition 3.3. For each $P \in C_0$, if $i(P) \neq 0$, then Theorem 2.5 and Lemma 2.8 imply

$$\begin{aligned}
& \psi(\{g_0(P), g_1(P), \dots, \partial_P(g_{i(P)}, f), \dots, g_{q-1}(P)\}_{k(P)/k}) \\
&= \text{Tr}_{k(P)/k}(g_0(P) \text{dlog}\{g_1(P), \dots, \partial_P(g_{i(P)}, f), \dots, g_{q-1}(P)\}) \\
&= \text{Tr}_{k(P)/k}(g_0(P) \text{dlog} \partial_P(\{g_1, \dots, g_{q-1}, f\})) \\
&= \text{Tr}_{k(P)/k}(g_0(P) \partial_P(\text{dlog } g_1, \dots, \text{dlog } g_{q-1} \text{dlog } f)) \\
&= \text{Tr}_{k(P)/k} \circ \partial_P(g_0 \text{dlog } g_1 \cdots \text{dlog } g_{q-1} \text{dlog } f).
\end{aligned}$$

If $i(P) = 0$, by the very definition, the equality $\partial_P(g_0 \text{dlog } f \text{dlog } g_i) = \partial_P(g_0 \text{dlog } f \text{dlog } g_i(P))$ holds. Hence we obtain

$$\begin{aligned}
& \psi(\{\partial_P(g_0, f), g_1(P), \dots, g_{q-1}(P)\}_{k(P)/k}) \\
&= \text{Tr}_{k(P)/k}(\partial_P(g_0 \text{dlog } f) \text{dlog } g_1(P) \cdots \text{dlog } g_{q-1}(P)) \\
&= \text{Tr}_{k(P)/k} \circ \partial_P(g_0 \text{dlog } g_1 \cdots \text{dlog } g_{q-1} \text{dlog } f).
\end{aligned}$$

From the trace formula (Thm. 2.6), we have

$$\psi\left(\sum_{P \in C_0} \{g_0(P), g_1(P), \dots, \partial_P(g_{i(P)}, f), \dots, g_{q-1}(P)\}_{k(P)/k}\right) = 0.$$

Thus ψ is well-defined.

Proof of the bijection: It is easy to see that $\psi \circ \phi$ is equal to the identity map on Ω_k^{q-1} . Hence to show the assertion it is enough to show that ϕ is surjective. Take an element $\{x, x_1, \dots, x_{q-1}\}_{E/k}$ in $K(k; \mathbb{G}_a, \mathbb{G}_m, \dots, \mathbb{G}_m)$. The trace map $\Omega_E^{q-1} \rightarrow \Omega_k^{q-1}$ is defined from the trace map $\text{Tr}_{E/k} : E \rightarrow k$ by identifying the isomorphism $E \otimes_k \Omega_k^{q-1} \simeq \Omega_E^{q-1}$ (Thm. 2.4). Therefore $\text{Tr}_{E/k}(x \text{dlog } x_1 \cdots \text{dlog } x_{q-1}) = \sum_i \text{Tr}_{E/k}(\xi_i) \text{dlog } z_{i,1} \cdots \text{dlog } z_{i,q-1}$ for some $\xi_i \in E$ and $z_{i,j} \in k^\times$. The property (PF) implies the equality

$$\{\text{Tr}_{E/k} \xi_i, z_{i,1}, \dots, z_{i,q-1}\}_{k/k} = \{\xi_i, z_{i,1}, \dots, z_{i,q-1}\}_{E/k}.$$

Therefore, defining $\phi_E : \Omega_E^{q-1} \rightarrow K(E; \mathbb{G}_a, \mathbb{G}_m, \dots, \mathbb{G}_m)$ by $\xi \text{dlog } \eta_1 \cdots \text{dlog } \eta_{q-1} \mapsto$

$\{\xi, \eta_1, \dots, \eta_{q-1}\}_{E/E}$ we have

$$\begin{aligned}
& \phi(\mathrm{Tr}_{E/k}(x \mathrm{dlog} x_1 \cdots \mathrm{dlog} x_{q-1})) \\
&= \sum_i \phi(\mathrm{Tr}_{E/k} \xi_i \mathrm{dlog} z_{i,1} \cdots \mathrm{dlog} z_{i,q-1}) \\
&= \sum_i N_{E/k} \circ \phi_E(\xi_i \mathrm{dlog} z_{i,1} \cdots \mathrm{dlog} z_{i,q-1}) \\
&= N_{E/k} \circ \phi_E(x \mathrm{dlog} x_1 \cdots \mathrm{dlog} x_{q-1}) \\
&= \{x, x_1, \dots, x_{q-1}\}_{E/k}.
\end{aligned}$$

Thus ϕ is surjective and we obtain the assertion. \square

The isomorphism $E \otimes_k \Omega_k^{q-1} \simeq \Omega_E^{q-1}$ appeared in the above proof is comparable to the following fact on the Milnor K -groups ([3], Chap. I, Cor. 5.3; see also the proof of Thm. 1.4 in [23]): For a prime number p , we assume that every finite extension of a field k is of degree p^r for some r . Then $K_q^M(E)$ is generated by elements of the form $\{x, y_1, \dots, y_{q-1}\}$, where $x \in E^\times$ and $y_i \in k^\times$.

If the characteristic of the field k is 0, we have $\mathbb{W}_S \Omega_k^{q-1} \simeq (\Omega_k^{q-1})^{\#S}$ (Prop. 2.3) and thus we obtain the following corollary.

Corollary 3.7. $K(k; \mathbb{W}_S, \overbrace{\mathbb{G}_m, \dots, \mathbb{G}_m}^{q-1}) \xrightarrow{\simeq} \mathbb{W}_S \Omega_k^{q-1}$.

Proof. The left becomes trivial if the field k has positive characteristic > 2 (Lem. 3.5). It is enough to show $\mathbb{W}_S \Omega_k^{q-1} = 0$ in this case. By Proposition 2.3 this reduces to the well-known result $W_m \Omega_k^{q-1} = 0$ for some m ([11], Chap. I, Prop. 1.6). \square

Formally, we can define the Somekawa K -group associated with plural number of Witt groups such as $K(k; \mathbb{W}_{S_1}, \dots, \mathbb{W}_{S_r}, G_1, \dots, G_q)$. However such group becomes trivial for $r > 1$ by the following observation.

Lemma 3.8. *For two finite truncation sets S and T , we have*

$$K(k; \mathbb{W}_S, \mathbb{W}_T) = 0.$$

Proof. From Proposition 2.3 and considering the restriction map from the product of the Witt group to the generalized Witt group, it is enough to show $K(k; W_m, W_n) = 0$ for some m and n . Any symbol $\{x, y\}_{E/k}$ in $K(k; W_m, W_n)$

is the image of the norm map $N_{E/k} : K(E; W_m, W_n) \rightarrow K(k; W_m, W_n)$ of $\{x, y\}_{E/E}$, hence we may assume $E = k$. Any element $x \in W_m(k)$ is written as $x = \sum_{i=0}^{m-1} V^i([x_i])$ for $x_i \in k$ (Lem. 2.2, (ii)). The assertion is now reduced to showing $\{V^i([x]), V^j([y])\}_{k/k} = 0$ for $x, y \in k$. Consider $K = k(t)$, $f = t \in K^\times$ and $g_1 = V^i([t]) \in W_m(K)$ and $g_2 = [t/(t-x)]V^j([y]) \in W_n(K)$. For the closed point $P = P_x$ corresponding to the prime ideal $(t-x)$, we have $g_1(P) = V^i([x])$ and $\partial_P(g_2 \operatorname{dlog}[f]) = V^j([y])$. For any closed point $P \neq P_x$, P_∞ , we have $\partial_P(g_2 \operatorname{dlog}[f]) = 0$. For the infinite place P_∞ ,

$$\begin{aligned} \partial_P(g_1 \operatorname{dlog}[f]) &= \partial_P(V^i([t])[t]^{-1}d[t]) \\ &= -\partial_P(V^i([t])[t]d[t^{-1}]) \\ &= -\partial_P(V^i([t]^{p^i+1})d[t^{-1}]) \\ &\stackrel{(*)}{=} 0. \end{aligned}$$

The last equality $(*)$ follows from Proposition 2.5, (f). Thus the Weil reciprocity law implies the assertion. \square

As noted before it is known that $G_1 \overset{M}{\otimes} \cdots \overset{M}{\otimes} G_q(k) = 0$ for semi-abelian varieties G_i if k is a finite field ([13]). However, even if k is a such finite field, the Mackey product $\mathbb{G}_a \overset{M}{\otimes} \mathbb{G}_a(k)$ is not trivial. In fact, we always have a surjective map $\mathbb{G}_a \overset{M}{\otimes} \mathbb{G}_a(k) \twoheadrightarrow k$ given by $\{x, y\}_{E/k} \mapsto \operatorname{Tr}_{E/k}(xy)$.

4 Mixed K -groups

For an equidimensional variety X over a perfect field k , let $\operatorname{CH}_0(X)$ be the Chow group of 0-cycles on X . Putting $X_E := X \otimes E$ for an extension E/k , the assignments $\mathcal{CH}_0(X) : E \mapsto \operatorname{CH}_0(X_E)$ give a Mackey functor by the pull-back $j^* : \operatorname{CH}_0(X_{E_2}) \rightarrow \operatorname{CH}_0(X_{E_1})$ and the push-forward $j_* : \operatorname{CH}_0(X_{E_1}) \rightarrow \operatorname{CH}_0(X_{E_2})$ for any finite field extension $j : E_1 \hookrightarrow E_2$ over k (cf. [8], Chap. 1).

Definition 4.1. Let $r, q \geq 0$ be integers, X_1, \dots, X_r smooth projective varieties over k and G_1, \dots, G_q split semi-abelian varieties over k . The *mixed K -group*

$$K(k; \mathbb{W}_S, G_1, \dots, G_q, \mathcal{CH}_0(X_1), \dots, \mathcal{CH}_0(X_r))$$

is defined by the quotient of the Mackey product

$$\left(\mathbb{W}_S \overset{M}{\otimes} G_1 \overset{M}{\otimes} \cdots \overset{M}{\otimes} G_q \overset{M}{\otimes} \mathcal{CH}_0(X_1) \overset{M}{\otimes} \cdots \overset{M}{\otimes} \mathcal{CH}_0(X_r)(k) \right) / R,$$

where R is the subgroup generated by the following elements:

(WR) Let $K = k(C)$ be the function field of a proper non-singular algebraic curve C over k . Put $G_0 := \mathbb{W}_S$. For $f \in K^\times, x_j \in \text{CH}_0((X_j)_K)$ and $g_i \in G_i(K)$, assume that for each closed point P in C there exists $i(P)$ such that $g_i \in G_i(\widehat{\mathcal{O}}_{C,P})$ for all $i \neq i(P)$. Then, the element is

$$\sum_{P \in C_0} g_0(P) \otimes \cdots \otimes \partial_P(g_{i(P)}, f) \otimes \cdots \otimes g_q(P) \otimes s_P(x_1) \otimes \cdots \otimes s_P(x_r).$$

Here $s_P : \text{CH}_0((X_j)_K) \rightarrow \text{CH}_0((X_j)_{k(P)})$ is the specialization map at P (cf. [8], Sect. 20.3).

This mixed K -group also has the structure of $\mathbb{W}_S(k)$ -module. From the following lemma, the mixed K -group defined above associated to varieties X_1, \dots, X_r over k is reduced to the one for the product $X_1 \times \cdots \times X_r$.

Lemma 4.2. *For any projective smooth varieties X and Y over k , we have*

$$K(k; \mathbb{W}_S, G_1, \dots, G_q, \mathcal{CH}_0(X), \mathcal{CH}_0(Y)) \xrightarrow{\cong} K(k; \mathbb{W}_S, G_1, \dots, G_q, \mathcal{CH}_0(X \times Y)).$$

Proof. We may assume that $q = 0$ and will show the isomorphism

$$(8) \quad \psi : K(k; \mathbb{W}_S, \mathcal{CH}_0(X), \mathcal{CH}_0(Y)) \xrightarrow{\cong} K(k; \mathbb{W}_S, \mathcal{CH}_0(X \times Y)).$$

Definition of ψ : By [18], Theorem 2.2, there is a canonical isomorphisms

$$\psi_k^{RS} : K(k; \mathcal{CH}_0(X), \mathcal{CH}_0(Y)) \xrightarrow{\cong} K(k; \mathcal{CH}_0(X \times Y)) \xrightarrow{\cong} \text{CH}_0(X \times Y)$$

defined by $\psi_k^{RS}(\{x, y\}_{E/k}) := N_{E/k}((p_X)^*x \cap (p_Y)^*y)$, where \cap is the intersection product, $p_X : X \times Y \rightarrow X, p_Y : X \times Y \rightarrow Y$ are projection maps, and $N_{E/k} : \text{CH}_0((X \times Y)_E) \rightarrow \text{CH}_0(X \times Y)$ is the push-forward on the Chow groups. We denote by ψ the composition

$$\mathbb{W}_S \otimes^M \mathcal{CH}_0(X) \otimes^M \mathcal{CH}_0(Y)(k) \twoheadrightarrow \mathbb{W}_S \otimes^M \mathcal{CH}_0(X \times Y)(k) \twoheadrightarrow K(k; \mathbb{W}_S, \mathcal{CH}_0(X \times Y)).$$

Precisely, on symbols, the map ψ is given by

$$\begin{aligned} \psi(\{w, x, y\}_{E/k}) &= \psi(\{w, \{x, y\}_{E/E}\}_{E/k}) \\ &= \{w, \psi_E^{RS}(\{x, y\}_{E/E})\}_{E/k} \\ &= \{w, (p_X)^*x \cap (p_Y)^*y\}_{E/k}. \end{aligned}$$

Now we show the map ψ vanishes on the elements of the form (WR) in the left of (8): $K(k; \mathcal{CH}_0(X), \mathcal{CH}_0(Y))$. Let $K = k(C)$ be the function field of a projective smooth curve C over k , $f \in K^\times$, $w \in \mathbb{W}_S(K)$, $x \in \text{CH}_0(X_K)$ and $y \in \text{CH}_0(Y_K)$. Since the specialization map is compatible with pull-backs ([8], Prop. 20.3 (a)), for any closed point $P \in C_0$, we have

$$\begin{aligned} \psi(\{\partial_P(w, f), s_P(x), s_P(y)\}_{k(P)/k}) &= \{\partial_P(w, f), (p_X)^* s_P(x) \cap (p_Y)^* s_P(y)\}_{k(P)/k} \\ &= \{\partial_P(w, f), s_P((p_X)^* x \cap (p_Y)^* (y))\}_{k(P)/k}. \end{aligned}$$

Thus (WR) in $K(k; \mathbb{W}_S, \mathcal{CH}_0(X \times Y))$ implies the assertion.

Definition of ϕ : Now we define

$$\phi : K(k; \mathbb{W}_S, \mathcal{CH}_0(X \times Y)) \longrightarrow K(k; \mathbb{W}_S, \mathcal{CH}_0(X), \mathcal{CH}_0(Y)).$$

For any finite field extension E/k , the Chow group $\mathcal{CH}_0((X \times Y)_E)$ is generated by closed points P in $(X \times Y)_E$. The map ϕ is defined for the closed points P in $(X \times Y)_E$ by $\phi(\{w, [P]\}_{E/k}) := \{w, (p_X)_*[P], (p_Y)_*[P]\}_{E/k}$. First we show this map is well-defined on the projection formula (PF): Let $j : E_1 \hookrightarrow E_2$ be a finite extensions over k . For $w \in \mathbb{W}_S(E_2)$ and a closed point P in $(X \times Y)_{E_1}$, we have

$$\begin{aligned} \phi(\{w, j^*[P]\}_{E_2/k}) &= \{w, (p_X)_* j^*[P], (p_Y)_* j^*[P]\}_{E_2/k} \\ &= \{w, j^*(p_X)_*[P], j^*(p_Y)_*[P]\}_{E_2/k} \\ &\stackrel{(*)}{=} \{\text{Tr}_{E_2/E_1} w, (p_X)_*[P], (p_Y)_*[P]\}_{E_1/k} \\ &= \phi(\{\text{Tr}_{E_2/E_1} w, [P]\}_{E_1/k}). \end{aligned}$$

For the equality (*), here we used the projection formula (PF) in the left of (8). Similarly, for $w \in \mathbb{W}_S(E_1)$ and a closed point P in $(X \times Y)_{E_2}$, we have

$$\begin{aligned} \phi(\{j^* w, [P]\}_{E_2/k}) &= \{j^* w, (p_X)_*[P], (p_Y)_*[P]\}_{E_2/k} \\ &= \{w, j_*(p_X)_*[P], j_*(p_Y)_*[P]\}_{E_1/k} \\ &= \{w, (p_X)_* j_*[P], (p_Y)_* j_*[P]\}_{E_1/k} \\ &= \phi(\{w, j_*[P]\}_{E_1/k}). \end{aligned}$$

Next, we show the map ϕ takes the elements of the form (WR) to zero in the target of (8). Let $K = k(C)$ be the function field of a projective smooth curve C over k , $f \in K^\times$, $w \in \mathbb{W}_S(K)$, ξ is a closed point in $(X \times Y)_K$. Since

the specialization map is compatible with push-forward ([8], Prop. 20.3 (b)), for any closed point $P \in C_0$, we have

$$\begin{aligned}\psi(\{\partial_P(w, f), s_P[\xi]\}_{k(P)/k}) &= \{\partial_P(w, f), (p_X)_* s_P[\xi], (p_Y)_* s_P[\xi]\}_{k(P)/k} \\ &= \{\partial_P(w, f), s_P \circ (p_X)_* x, s_P \circ (p_Y)_* [\xi]\}_{k(P)/k}.\end{aligned}$$

Thus the assertion follows from (WR) in $K(k; \mathbb{W}_S, \mathcal{CH}_0(X), \mathcal{CH}_0(Y))$.

Proof of the bijection: For any symbol $\{w, [P_X], [P_Y]\}_{E/k}$ in the left of (8) given by closed points P_X in X_E and P_Y in Y_E , put $E(P_X, P_Y) = E(P)$, where P is a closed point in $(X \times Y)_E$ determined by P_X and P_Y , the projection formula (PF) implies

$$\begin{aligned}\{w, [P_X], [P_Y]\}_{E/k} &= \{\mathrm{Tr}_{E(P)/E} \tilde{w}, [P_X], [P_Y]\}_{E/k} \\ &= \{\tilde{w}, j^*[P_X], [P_Y]\}_{E(P)/k}\end{aligned}$$

for some $\tilde{w} \in \mathbb{W}_S(E(P))$ and $j : E \hookrightarrow E(P)$. Hence $K(k; \mathbb{W}_S, \mathcal{CH}_0(X), \mathcal{CH}_0(Y))$ is generated by symbols of the form $\{w, P_X, P_Y\}_{E/k}$, where P_X and P_Y are E -rational points. The same holds in $K(k; \mathcal{CH}_0(X \times Y))$. Now it is easy to see that ϕ is surjective and $\psi \circ \phi(\{w, [P]\}_{E/k}) = \{w, [P]\}_{E/k}$ for E -rational point P in $(X \times Y)_E$. and the bijectivity follows from it. \square

Lemma 4.3. (i) *If $X = \mathrm{Spec} k$, then*

$$K(k; \mathbb{W}_S, \mathcal{CH}_0(\mathrm{Spec} k)) \xrightarrow{\cong} K(k; \mathbb{W}_S) \simeq \mathbb{W}_S(k).$$

(ii) *Let k be a perfect field of characteristic $p > 2$. If X is a projective smooth curve over k with $X(k) \neq \emptyset$, then*

$$K(k; \mathbb{W}_S, \mathcal{CH}_0(X)) \xrightarrow{\cong} K(k; \mathbb{W}_S) \simeq \mathbb{W}_S(k).$$

Proof. The assertion (i) follows from $\mathrm{CH}_0(\mathrm{Spec} E) \simeq \mathbb{Z}$ for an extension E/k . For the assertion (ii), we consider the following split exact sequence

$$0 \rightarrow A_0(X) \rightarrow \mathrm{CH}_0(X) \xrightarrow{\deg} \mathbb{Z} \rightarrow 0,$$

where \deg is the degree map and $A_0(X)$ is defined by the exactness. By replacing CH_0 with A_0 in the appropriate instances, we can define the mixed K -group $K(k; \mathbb{W}_S, \mathcal{A}_0(X))$ as a quotient of $\mathbb{W}_S \otimes^M \mathcal{A}_0(X)(k)$. Recall that \mathbb{Z}

is the unit of the Mackey product. The above sequence gives a split short exact sequence

$$0 \rightarrow K(k; \mathbb{W}_S, \mathcal{A}_0(X)) \rightarrow K(k; \mathbb{W}_S, \mathcal{CH}_0(X)) \rightarrow K(k; \mathbb{W}_S) \rightarrow 0.$$

Because of $K(k; \mathbb{W}_S) \simeq \mathbb{W}_S(k)$ (Lem. 3.4), it is enough to show $\mathbb{W}_S \otimes^M \mathcal{A}_0(X)(k) = 0$. By the assumption $X(k) \neq \emptyset$, we have $\mathcal{A}_0(X) \simeq J_X(k)$ as Mackey functors, where J_X is the Jacobian variety of X . Therefore we obtain $\mathbb{W}_S \otimes^M \mathcal{A}_0(X)(k) \simeq \mathbb{W}_S \otimes^M J_X(k)$, and the latter becomes trivial by Lemma 3.5. \square

Let X be a projective smooth curve over k . Consider the complex

$$K_2(k(X)) \xrightarrow{\partial} \bigoplus_{P \in X_0} k(P)^\times \xrightarrow{N} k^\times$$

where ∂ and N are defined by the tame symbols (5) as $\partial := \bigoplus_P \partial_P$ and the norm maps $N := \sum_P N_{k(P)/k}$ respectively. Here, the fact $N \circ \partial = 0$ is called the Weil reciprocity law on the Milnor K -groups (*cf.* [3], Sect. 5). We denote by $V(X) := \text{Ker}(N)/\text{Im}(\partial)$ the homology group of the above complex.

Theorem 4.4 ([23], Thm. 2.1, see also [2]). *There is a canonical isomorphism*

$$K(k; \mathbb{G}_m, \mathcal{CH}_0(X)) \xrightarrow{\simeq} \text{Coker} \left(\partial : K_2(k(X)) \rightarrow \bigoplus_{P \in X_0} k(P)^\times \right).$$

Note that the cokernel on the right is often denoted by $SK_1(X)$. In particular, if we further assume X has a k -rational point, then the decomposition $A_0(X) \oplus \mathbb{Z} \simeq \text{CH}_0(X)$ gives an isomorphism $K(k, \mathbb{G}_m, J_X) \simeq V(X)$, where J_X is the Jacobian variety of X .

Now, we study the analogy of the above theorem. For the projective smooth curve X over k , we consider the following complex

$$(9) \quad \mathbb{W}_S \Omega_{k(X)}^1 \xrightarrow{\partial} \bigoplus_{P \in X_0} \mathbb{W}_S(k(P)) \xrightarrow{\text{Tr}} \mathbb{W}_S(k)$$

where ∂ and Tr are defined by the residue maps $\partial := \bigoplus_P \partial_P$ and the trace maps $\text{Tr} := \sum_P \text{Tr}_{k(P)/k}$ respectively (*cf.* Thm. 2.6).

Theorem 4.5. *There is a canonical isomorphism*

$$\psi : K(k; \mathbb{G}_a, \mathcal{CH}_0(X)) \xrightarrow{\cong} \text{Coker} \left(\partial : \Omega_{k(X)}^1 \rightarrow \bigoplus_{P \in X_0} k(P) \right).$$

If we further assume $X(k) \neq \emptyset$, putting $V(X)^+ := \text{Ker}(\text{Tr}) / \text{Im}(\partial)$ we also obtain $K(k; \mathbb{G}_a, J_X) \simeq V(X)^+$.

Since the product of Quillen's K -theory was used in Somekawa's proof, our proof is based on Akhtar's rather direct proof in her thesis ([1], Thm. 6.2.1).

Proof of Thm. 4.5. Definition of ϕ : A map

$$\phi : \text{Coker}(\partial) \rightarrow K(k; \mathbb{G}_a, \mathcal{CH}_0(X))$$

is defined by $[\sum_P x_P] \mapsto \sum_P \{x_P, j_P^*[P]\}_{k(P)/k}$. Here $[\sum_P x_P]$ is the class in $\text{Coker}(\partial)$ represented by $\sum_P x_P \in \bigoplus_P k(P)$ and $[P]$ is the cycle in $\text{CH}_0(X)$ represented by the point P and $j_P^* : \text{CH}_0(X) \rightarrow \text{CH}_0(X_{k(P)})$ is the pullback along (the base change to X of) $j_P = j_{k(P)/k} : k \hookrightarrow k(P)$. We show that this correspondence is well-defined. For each closed point P in X , we have $s_P([\xi]) = j_P^*[P]$, where $[\xi] \in \text{CH}_0(X_{k(X)})$ is the cycle on $X_{k(X)}$ arising from the generic point ξ in X , ([8], Sect. 20.3) For any $f \text{ dlog } g \in \Omega_{k(X)}^1$ ($f \in k(X)$ and $g \in k(X)^\times$),

$$\begin{aligned} \phi(\partial_P(f \text{ dlog } g)) &= \{\partial_P(f \text{ dlog } g), j_P^*[P]\}_{k(P)/k} \\ &= \{\partial_P(f, g), s_P([\xi])\}_{k(P)/k}. \end{aligned}$$

Therefore the condition (WR) on the mixed K -group (Def. 4.1) asserts that the map ϕ is well-defined.

Definition of ψ : The map

$$\psi : K(k; \mathbb{G}_a, \mathcal{CH}_0(X)) \rightarrow \text{Coker}(\partial)$$

is given from the following diagram (cf. [4], Sect. 1): For each finite extension

E/k ,

$$\begin{array}{ccccccc}
E \otimes E(X)^\times & \longrightarrow & \bigoplus_{Q \in (X_E)_0} E & \longrightarrow & E \otimes \mathrm{CH}_0(X_E) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\Omega_{E(X)}^1 & \xrightarrow{\partial_E} & \bigoplus_{Q \in (X_E)_0} E(Q) & \longrightarrow & \mathrm{Coker}(\partial_E) & \longrightarrow & 0 \\
\mathrm{Tr}_{E(X)/k(X)} \downarrow & & \downarrow & & \downarrow & & \\
\Omega_{k(X)}^1 & \xrightarrow{\partial} & \bigoplus_{P \in X_0} k(P) & \longrightarrow & \mathrm{Coker}(\partial) & \longrightarrow & 0.
\end{array}$$

Here the left vertical map $E \otimes E(X)^\times \rightarrow \Omega_{E(X)}^1$ is given by $x \otimes f \mapsto x \, \mathrm{dlog} \, f$ and the commutativity follows from Lemma 4.6. More precisely, for $x \in E$ and $Q \in (X_E)_0$, we have

$$\psi(\{x, [Q]\}_{E/k}) = \mathrm{Tr}_{E(Q)/k(P)}(x) \quad \text{at } P \in X_0,$$

where $P = j_{E/k}(Q)$ the image of Q by $j_{E/k} : X_E \rightarrow X$. We show that ψ is well-defined. For any finite extension $j = j_{E_1/E_2} : E_1 \hookrightarrow E_2$ over k , take $x \in E_1$ and a closed point Q_2 in X_{E_2} . Then, $j_*[Q_2] = [E_2(Q_2) : E_1(Q_1)][Q_1]$, where Q_1 is the image of Q_2 in X_{E_1} . We obtain the following equalities

$$\begin{aligned}
\psi(\{x, [Q_2]\}_{E_2/k}) &= \mathrm{Tr}_{E_2(Q_2)/k(P)}(x) \\
&= [E_2(Q_2) : E_1(Q_1)] \mathrm{Tr}_{E_1(Q_1)/k(P)}(x) \\
&= \psi(\{x, j_*[Q_2]\}).
\end{aligned}$$

On the other hand, we assume that $x \in E_2$, $Q_1 \in (X_{E_1})_0$. Here the pull-back $j^*[Q_1]$ is defined to be the fundamental cycle $[j^{-1}(Q_1)]$ of the inverse image $j^{-1}(Q_1)$. Denoting by $[j^{-1}(Q_1)] = \sum_Q m_Q [Q] \in \mathrm{CH}_0(X_{E_2})$ and $P = j_{E_1/k}(Q_1)$, we have

$$\begin{aligned}
\psi(\{x, j^*[Q_1]\}_{E_2/k}) &= \sum_Q m_Q \psi(\{x, [Q]\}_{E_2/k}) \\
&= \sum_Q m_Q \mathrm{Tr}_{E_2(Q)/k(P)}(x) \\
&= \mathrm{Tr}_{E_1(Q_1)/k(P)} \circ \mathrm{Tr}_{E_2/E_1}(x) \\
&= \psi(\{\mathrm{Tr}_{E_2/E_1}(x), [Q_1]\}_{E_1/k}).
\end{aligned}$$

As a result, the map ψ vanishes the elements of the form (PF) and this gives $\psi : \mathbb{G}_a^{\otimes M} \otimes \mathcal{CH}_0(X)(k) \rightarrow \mathrm{Coker}(\partial)$. Next we consider the relation (WR). Let

$K = k(C)$ be the function field of a projective smooth curve C over k . For $f \in K^\times, g \in K$ and ξ a closed point in X_K , our claim is now

$$(10) \quad \sum_{Q \in C_0} \psi(\{\partial_Q(g, f), s_Q([\xi])\}_{k(Q)/k}) = 0.$$

To show this claim, first we assume that $K(\xi) = K$, that is the given closed point ξ is a K -rational point of X_K . Under this assumption, for each closed point Q in C , the valuative criterion for properness gives the diagram

$$\begin{array}{ccccc} & & \text{Spec } K & \xrightarrow{\xi} & X_K & \longrightarrow & X \\ & & \downarrow & \nearrow & & & \downarrow \\ \text{Spec } k(Q) & \longrightarrow & \text{Spec } \mathcal{O}_{C,Q} & \longrightarrow & \text{Spec } k. & & \end{array}$$

The composition $\text{Spec } K \xrightarrow{\xi} X_K \rightarrow X$ extends uniquely to $\pi : C \rightarrow X$. The specialization is now $s_Q([\xi]) = (j_Q)^*[P]$ in $\text{CH}_0(X_{k(Q)})$, where $P = \pi(Q)$ and $j_Q : k \hookrightarrow k(Q)$ (cf. [18], (2.2.2)). If the image of ξ in X gives a closed point P_0 in X , then π is a constant map; $P = \pi(Q) = P_0$ for any $Q \in C_0$ and thus the curve C is defined over $k(P_0)$. We obtain

$$\begin{aligned} & \sum_{Q \in C_0} \psi(\{\partial_Q(g, f), s_Q([\xi])\}_{k(Q)/k}) \\ &= \sum_{Q \in C_0} \psi(\{\partial_Q(g, f), (j_Q)^*[P]\}_{k(Q)/k}) \\ &= \left(\sum_{Q \in C_0} \text{Tr}_{k(Q)/k(P_0)} \partial_Q(g \, \text{dlog } f) \right) \text{ at } P_0 \in X_0. \end{aligned}$$

The trace formula for C over $k(P_0)$ (Thm. 2.6) gives the claim. On the other

hand, when the image of ξ is the generic point of X , we have

$$\begin{aligned}
& \sum_{Q \in C_0} \psi(\{\partial_Q(g, f), s_Q([\xi])\}_{k(Q)/k}) \\
&= \sum_{Q \in C_0} \psi(\{\partial_Q(g, f), (j_Q)^*[P]\}_{k(Q)/k}) \\
&= \sum_{Q \in C_0} (\text{Tr}_{k(Q)/k(P)} \circ \partial_Q(g \text{ dlog } f) \text{ at } P = \pi(Q) \in X_0) \\
&= \sum_{P \in X_0} \sum_{Q \in \pi^{-1}(P)} \text{Tr}_{k(Q)/k(P)} \circ \partial_Q(g \text{ dlog } f) \\
&\stackrel{(*)}{=} \sum_{P \in X_0} \partial_P \circ \text{Tr}_{k(C)/k(X)}(g \text{ dlog } f) \\
&= \partial \circ \text{Tr}_{k(C)/k(X)}(g \text{ dlog } f) \\
&= 0 \quad \text{in } \text{Coker}(\partial).
\end{aligned}$$

Here, the equality $(*)$ follows from Lemma 4.6 below.

Next, we show the claim for arbitrary $\xi \in (X_K)_0$. Then $K' := K(\xi)$ is a finite extension over K . Let C' be the regular proper model of K' . The specialization map on the Chow groups has the following functorial property: For each $Q \in C_0$,

$$\begin{array}{ccc}
\text{CH}_0(X_{K'}) & \xrightarrow{e(Q'/Q)s_{Q'}} & \bigoplus_{Q' \in C'_0, Q'|Q} \text{CH}_0(X_{k(Q')}) \\
(j_{K'/K})_* \downarrow & & \downarrow \sum (j_{k(Q')/k(Q)})_* \\
\text{CH}_0(X_K) & \xrightarrow{s_Q} & \text{CH}_0(X_{k(Q)}),
\end{array}$$

where $e(Q'/Q)$ is the ramification index of Q' over Q ([8], Sect. 20.3). Denoting by ξ' the closed point in $X_{K'}$ which is determined by ξ , we have

$$s_Q([\xi]) = s_Q \circ (j_{K'/K})_*([\xi']) = \sum_{Q'|Q} e(Q'/Q) j_{k(Q')/k(Q)} \circ s_{Q'}([\xi']).$$

The projection formula (PF) and Lemma 2.9 implies

$$\begin{aligned}
& \sum_{Q \in C_0} \{\partial_Q(g, f), s_Q([\xi])\}_{k(Q)/k} \\
&= \sum_{Q \in C_0} \sum_{Q' | Q} e(Q'/Q) \{\partial_Q(g, f), (j_{k(Q')/k(Q)})_* s_{Q'}([\xi'])\}_{k(Q)/k} \\
&= \sum_{Q' \in C'_0} \{e(Q'/Q) \partial_Q(g, f), s_{Q'}([\xi'])\}_{k(Q')/k} \\
&= \sum_{Q' \in C'_0} \{\partial_{Q'}(g, f), s_{Q'}([\xi'])\}_{k(Q')/k}.
\end{aligned}$$

Therefore, the claim is now reduced to the case treated before.

Proof of the bijection: It is easy to see that $\psi \circ \phi = \text{Id}$ and thus we show the map ϕ is surjective. For any symbol $\{x, [Q]\}_{E/k}$ with a closed point Q of X_E , take $y \in E(Q)$ such that $\text{Tr}_{E(Q)/E}(y) = x$. By the projection formula (PF), we have $\{x, [Q]\}_{E/k} = \{y, j_{E(Q)/E}^*[Q]\}_{E(Q)/E}$. Hence we reduce to the case $E(Q) = E$. In this case $E(Q) = E$, denoting $P := j_{E/k}(Q)$ the closed point in X determined by the image of $j_{E/k} : X_E \rightarrow X$, the surjectivity follows from the following equalities

$$\begin{aligned}
\phi([\text{Tr}_{E/k(P)}(x)]) &= \{\text{Tr}_{E/k(P)}(x), j_P^*[P]\}_{k(P)/k} \\
&= \{x, j_{E/k(P)}^* \circ j_P^*[P]\}_{E/k}.
\end{aligned}$$

□

With slight modification of the proof of the trace formula ([16] Thm. 17.6) we show that the residue map in our sense commutes with the trace maps.

Lemma 4.6. *Let $C' \rightarrow C$ be a dominant morphism of smooth projective curves over k . If we fix a closed point P of C , we have the following commutative diagram:*

$$\begin{array}{ccc}
\Omega_{k(C')}^1 & \xrightarrow{\oplus \partial_{P'}} & \bigoplus_{P' | P} k(P') \\
\text{Tr}_{k(C')/k(C)} \downarrow & & \downarrow \sum \text{Tr}_{k(P')/k(P)} \\
\Omega_{k(C)}^1 & \xrightarrow{\partial_P} & k(P),
\end{array}$$

where P' runs through all points in C' lies above P .

Proof. Considering the completion $k(C')_{P'}$ and $k(C)_P$ we have

$$\begin{array}{ccc} \Omega_{k(C')}^1 & \longrightarrow & \Omega_{k(C')_{P'}}^1 \\ \text{Tr}_{k(C')/k(C)} \downarrow & & \downarrow \text{Tr}_{k(C')_{P'}/k(C)_P} \\ \Omega_{k(C)}^1 & \longrightarrow & \Omega_{k(C)_P}^1. \end{array}$$

This carries over to the local situation. Let k'/k be a finite separable extension and consider the local fields $K' = k'((t'))$ and $K = k((t))$. Now $K^{\text{ur}} = k'((t))$ is the maximal unramified extension in K' over K . Since any element $\omega \in \Omega_{K^{\text{ur}}}^1$ is written as

$$\omega = \sum_{j \in \mathbb{Z}} a_j t^j + b_j t^{j-1} dt.$$

for some $a_j \in \Omega_{k'}^1$, $b_j \in k'$, we have

$$\partial_K(\text{Tr}_{K^{\text{ur}}/K}(\omega)) = \text{Tr}_{k'/k}(b_0) = \text{Tr}_{k'/k} \circ \partial_{K^{\text{ur}}}(\omega).$$

Thus we may assume that $k = k'$ and hence $K^{\text{ur}} = K$. We denote by \bar{k} an algebraic closure of k . By the very definition, for any $\omega \in \Omega_{K'}^1$, we have $\text{Res}_{K'}(\omega) = \text{Res}_{\bar{k}(t')}(\omega)$ and the trace map is compatible with base change $\text{Tr}_{\bar{k}(t')/\bar{k}(t)}(\omega) = \text{Tr}_{k(t')/k(t)}(\omega)$. We further reduce to the case $k = \bar{k}$. In this case the residue map ∂_K and the residue map Res_K coincide and the required assertion is well-known ([16], Thm. 17.6). \square

For any finite truncation set S , by the same manner as in the above proof we can show the existence of the surjection

$$\psi : \mathbb{W}_S \otimes^M \mathcal{CH}_0(X)(k) \longrightarrow \text{Coker} \left(\partial : \mathbb{W}_S \Omega_{k(X)}^1 \rightarrow \bigoplus_{P \in X_0} \mathbb{W}_S(k(P)) \right)$$

using Lemma 4.6. Denoting by $V_S(X)^+ := \text{Ker}(\text{Tr})/\text{Im}(\partial)$ as above, we obtain

$$\begin{array}{ccccccc} \mathbb{W}_S \otimes^M \mathcal{A}_0(X)(k) & \longrightarrow & \mathbb{W}_S \otimes^M \mathcal{CH}_0(X)(k) & \longrightarrow & \mathbb{W}_S(k) & \longrightarrow & 0 \\ \downarrow & & \downarrow \psi & & \parallel & & \\ 0 & \longrightarrow & V_S(X)^+ & \longrightarrow & \text{Coker}(\partial) & \xrightarrow{\text{Tr}} & \mathbb{W}_S(k) \longrightarrow 0. \end{array}$$

From Lemma 4.3, we obtain $\mathbb{W}_S \overset{M}{\otimes} \mathcal{A}_0(X) = 0$ if k has positive characteristic and this gives the following corollary:

Corollary 4.7. *If k has positive characteristic > 2 , then we have $V_S(X)^+ = 0$. In other words, the sequence (9) is exact.*

5 (Additive) higher Chow groups of schemes

Fix a field k as a base field. Let X be a scheme of finite type over k . Set $\square^q := (\mathbb{P}^1 \setminus \{1\})^q$ and we use the coordinates (y_1, \dots, y_q) on \square^q . The subscheme of \square^q defined by equations $y_{i_1} = \varepsilon_1, \dots, y_{i_s} = \varepsilon_s$ for $\varepsilon_j \in \{0, \infty\}$ is called a *face* of \square^q . For $\varepsilon \in \{0, \infty\}$ and $i = 1, \dots, q-1$, let $\iota_{q,i,\varepsilon} : \square^{q-1} \rightarrow \square^q$ be the inclusion defined by $(y_1, \dots, y_{q-1}) \mapsto (y_1, \dots, y_{i-1}, \varepsilon, y_i, \dots, y_{q-1})$. For an integer $q \geq 0$, let $\mathbb{Z}_p(X, q)$ be the free abelian group on integral closed subschemes Z of $X \times \square^q$ of dimension $p + q$ that intersect all faces of \square^q properly. For each i and ε , let $\partial_i^\varepsilon := \text{Id}_X \times \iota_{q,i,\varepsilon}^*$. The above condition assures that $\partial_i^\varepsilon(Z)$ is in $\mathbb{Z}_p(X, q-1)$, for a generator Z in $\mathbb{Z}_p(X, q)$ and $\mathbb{Z}_p(X, *)$ is a complex with boundary map

$$\sum_{i=1}^q (-1)^i (\partial_i^\infty - \partial_i^0) : \mathbb{Z}_p(X, q) \rightarrow \mathbb{Z}_p(X, q-1).$$

The *higher Chow complex* $\mathbb{Z}_p(X, *)$ is $\mathbb{Z}_p(X, *)$ modulo the complex consists of the degenerate cycles, that is the cycles on $X \times \square^q$ pulled back from cycles on $X \times \square^{q-1}$ by a projection $X \times \square^q \rightarrow X \times \square^{q-1}$ of the form $(x, y_1, \dots, y_q) \mapsto (x, y_1, \dots, y_{k-1}, y_{k+1}, \dots, y_q)$ for some k . The homology group

$$\text{CH}_p(X, q) := H_q(\mathbb{Z}_p(X, *))$$

is called the *higher Chow group* of X . If the scheme X is equidimensional of dimension d over k , we write $\mathbb{Z}^p(X, q) := \mathbb{Z}_{d-p}(X, q)$ and $\text{CH}^p(X, q) := H_q(\mathbb{Z}^p(X, *))$. The higher Chow groups have a functorial properties induced from the proper push-forward, and the flat pull-back of cycles. For two schemes X, Y of finite type over k , one can construct the external product

$$\boxtimes : \mathbb{Z}_p(X, *) \otimes \mathbb{Z}_r(Y, *) \rightarrow \mathbb{Z}_{p+r}(X \times Y, *).$$

On integral cycles it is defined by $Z \boxtimes W := \tau_*(Z \times W)$ where $\tau : X \times \square^p \times Y \times \square^r \rightarrow X \times Y \times \square^{p+r}$ is the exchange of factors. On homology groups,

\boxtimes induces the external product

$$(11) \quad \boxtimes : \mathrm{CH}_p(X, q) \otimes \mathrm{CH}_r(Y, s) \rightarrow \mathrm{CH}_{p+r}(X \times Y, q + s).$$

Here we list some calculations of higher Chow groups. First there is a natural isomorphism $\mathrm{CH}^p(X, 0) \simeq \mathrm{CH}^p(X)$ where the later is the ordinary Chow group. Put $\mathrm{CH}^p(k, q) := \mathrm{CH}^p(\mathrm{Spec}(k), q)$ by convention. In this case we have

Theorem 5.1 ([17], [24]).

$$\mathrm{CH}^p(k, q) \simeq \begin{cases} 0, & p > q, \\ K_q^M(k), & p = q. \end{cases}$$

On 0-cycles the results relevant here are the following:

Theorem 5.2 ([2], Cor. 7.1). *If X is a smooth quasi-projective variety over a finite field k of dimension d , we have $\mathrm{CH}^{d+q}(X, q) = 0$ for $q \geq 2$.*

Next we recall the additive higher Chow groups. Let

$$B_q := \mathbb{A}^1 \times \square^{q-1} \subset \overline{B}_q := \mathbb{A}^1 \times (\mathbb{P}^1)^{q-1}.$$

We use the coordinates (t, y_1, \dots, y_{q-1}) on \overline{B}_q . The subscheme of B_q defined by equations $y_{i_1} = \varepsilon_1, \dots, y_{i_s} = \varepsilon_s$ for $\varepsilon_j \in \{0, \infty\}$ is called a *face* of B_q . For $\varepsilon \in \{0, \infty\}$ and $i = 1, \dots, q-1$, let $\iota_{q,i,\varepsilon} : B_{q-1} \rightarrow B_q$ be the inclusion defined by $(t, y_1, \dots, y_{q-2}) \mapsto (t, y_1, \dots, y_{i-1}, \varepsilon, y_i, \dots, y_{q-2})$. On \overline{B}_q let $F_{q,i}^1$ be the Cartier divisor defined by $y_i = 1$ and $F_{q,0}$ the Cartier divisor defined by $t = 0$. Let m be a positive integer. Define $\underline{\mathrm{Tz}}_p(X, 1; m)$ to be the free abelian group on integral closed subschemes Z of $X \times \mathbb{A}^1$ of dimension p such that $Z \cap (X \times \{0\}) = \emptyset$. For an integer $q > 1$, $\underline{\mathrm{Tz}}_p(X, q; m)$ is the free abelian group on integral closed subschemes Z of $X \times B_q$ of dimension $p + q - 1$ satisfying the following conditions:

(Good position) For each face F of B_q , Z intersects $X \times F$ properly.

(Modulus condition) Let $\pi : \overline{Z}^N \rightarrow X \times \overline{B}_q$ be the normalization of the closure of Z in $X \times \overline{B}_q$. Then

$$(m+1)\pi^*(X \times F_{q,0}) \leq \sup_{1 \leq i \leq q-1} \pi^*(X \times F_{q,i}^1)$$

as Weil divisors. Here, for Weil divisors D_1, \dots, D_q on a normal variety, $\sup_i D_i$ is the Weil divisor defined by

$$\sup_{1 \leq i \leq q} D_i := \sum_{V: \text{prime divisor}} \left(\max_{1 \leq i \leq q} \text{ord}_V(D_i) \right) [V].$$

(For the other similar conditions on modulus, see [15]).

For each i and ε , let $\partial_i^\varepsilon := \text{Id}_X \times \iota_{q,i,\varepsilon}^*$. The above conditions assure $\partial_i^\varepsilon(Z)$ is in $\underline{\text{Tz}}_p(X, q-1; m)$, for a generator Z in $\underline{\text{Tz}}_p(X, q; m)$. The boundary map is given by

$$\sum_{i=1}^{q-1} (-1)^i (\partial_i^\infty - \partial_i^0) : \underline{\text{Tz}}_p(X, q; m) \rightarrow \underline{\text{Tz}}_p(X, q-1; m).$$

The *additive cycle complex* $\text{Tz}_p(X, *, m)$ is the nondegenerate complex associated to $\underline{\text{Tz}}_p(X, *, m)$. Its homology group

$$\text{TCH}_p(X, q; m) := H_q(\text{Tz}_p(X, *, m))$$

is called the *additive higher Chow group* of X with modulus m . If the scheme X is equidimensional of dimension d over k , we write $\text{Tz}^p(X, q; m) := \text{Tz}_{d+1-p}(X, q; m)$ and $\text{TCH}^p(X, q; m) := H_q(\text{Tz}^p(X, *, m))$. The additive higher Chow groups have a functorial properties as projective push-forward, and the flat pull-back. For two equidimensional schemes X, Y of finite type over k , one can construct the external product

$$\boxtimes : \text{z}_p(X, *) \otimes \text{Tz}_r(Y, *, m) \rightarrow \text{Tz}_{p+r}(X \times Y, *, m).$$

On integral cycles it is defined by $Z \boxtimes W := \tau_*(Z \times W)$ where $\tau : X \times \square^p \times Y \times B_r \rightarrow X \times Y \times B_{p+r}$ is the exchange of factors. On homology groups, \boxtimes induces the external product

$$\boxtimes : \text{CH}_p(X, q) \otimes \text{TCH}_r(Y, s) \rightarrow \text{TCH}_{p+r}(X \times Y, q + s).$$

For the case $X = Y$ and if we assume that X is smooth and projective variety over k , by composing the pullback along the diagonal map (more precisely, see [14], Thm. 4.10) we obtain the intersection product

$$\cap : \text{CH}_p(X, q) \otimes \text{TCH}_r(X, s; m) \rightarrow \text{TCH}_{p+r}(X, q + s; m)$$

which is compatible with flat pull-back and satisfying the projection formula: $f_*(f^*(x) \cap y) = x \cap f_*(y)$ for $f : X \rightarrow Y$ a morphism of smooth projective varieties over k . If f is flat, we also have $f_*(x \cap f^*(y)) = f_*(x) \cap y$. Putting $\mathrm{TCH}^p(k, q; m) := \mathrm{TCH}^p(\mathrm{Spec}(k), q; m)$ by convention we also have the following theorem.

Theorem 5.3 ([19]).

$$\mathrm{TCH}^p(k, q; m) \simeq \begin{cases} 0, & p > q, \\ \mathbb{W}_m \Omega_k^{q-1}, & p = q \end{cases}$$

if the field k has characteristic $\neq 2$.

For an equidimensional variety X over k , putting $X_E := X \otimes E$ the assignments

$$\mathcal{CH}^p(X, q) : E \mapsto \mathrm{CH}^p(X_E, q) \quad \text{and} \quad \mathcal{TCH}^p(X, q; m) : E \mapsto \mathrm{TCH}^p(X_E, q; m)$$

give Mackey functors respectively. Let a, a_1, \dots, a_q be integers, b, m positive integers and b_1, \dots, b_q non-negative integers. For smooth projective varieties X, X_1, \dots, X_q over a field k , we consider the product $\mathcal{TCH}^a(X, b; m) \otimes^M \mathcal{CH}^{a_1}(X_1, b_1) \otimes^M \dots \otimes^M \mathcal{CH}^{a_q}(X_q, b_q)(k)$ as Mackey functors. For any finite field extension E/k , let $\alpha := a + \sum_i a_i$ and $\beta := b + \sum_i b_i$. We define a homomorphism

$$\begin{aligned} \psi : \mathcal{TCH}^a(X, b; m) \otimes^M \mathcal{CH}^{a_1}(X_1, b_1) \otimes^M \dots \otimes^M \mathcal{CH}^{a_q}(X_q, b_q)(k) \\ \longrightarrow \mathrm{TCH}^\alpha(X \times X_1 \times \dots \times X_q, \beta; m) \end{aligned}$$

by the intersection products

$$\psi(\{x, x_1, \dots, x_q\}_{E/k}) := \mathrm{Tr}_{E/k}(p_1^*(x_1) \cap \dots \cap p_q^*(x_q) \cap p^*(x)),$$

where $\mathrm{Tr}_{E/k} := j_*$ is the pullback along $j : \mathrm{Spec}(E) \rightarrow \mathrm{Spec}(k)$ and p and p_i are the base change of the projection from $X \times X_1 \times \dots \times X_q$ to X and X_i respectively. From the projection formula, the map ψ is well-defined. We show the surjectivity of ψ on 0-cycles:

Theorem 5.4. *Let d and d_i be the dimension of X and X_i respectively. The map ψ is surjective on $a = d + b$ and $a_i = d_i + b_i$.*

Proof. By [18] (2.4.4), the intersection product as above gives an isomorphism

$$K(k; \mathcal{CH}^{d_1+b_1}(X_1, b_1), \mathcal{CH}^{d_2+b_2}(X_2, b_2)) \xrightarrow{\simeq} CH^{d_1+d_2+b_1+b_2}(X_1 \times X_2, b_1 + b_2).$$

Thus the composition with the canonical map

$$\mathcal{CH}^{b_1+d_1}(X_1, b_1) \overset{M}{\otimes} \mathcal{CH}^{d_2+b_2}(X_2, b_2)(k) \twoheadrightarrow K(k; \mathcal{CH}^{d_1+b_1}(X_1, b_1), \mathcal{CH}^{d_2+b_2}(X_2, b_2))$$

induces a surjection

$$\begin{aligned} \mathcal{TC}\mathcal{H}^{d+b}(X, b) \overset{M}{\otimes} \mathcal{CH}^{b_1+d_1}(X_1, b_1) \overset{M}{\otimes} \mathcal{CH}^{d_2+b_2}(X_2, b_2)(k) \\ \longrightarrow \mathcal{TC}\mathcal{H}^{d+b}(X, b) \overset{M}{\otimes} \mathcal{CH}^{d_1+d_2+b_1+b_2}(X_1 \times X_2, b_1 + b_2)(k). \end{aligned}$$

Hence it is enough to show the case $q = 1$: the surjection

$$\psi : \mathcal{TC}\mathcal{H}^{d+b}(X, b) \overset{M}{\otimes} \mathcal{CH}^{b'+d'}(X', b')(k) \twoheadrightarrow \mathrm{TCH}^{\delta+\beta}(X \times X', \beta),$$

for projective smooth variety X' over k of dimension d' , where $\delta := d + d'$ and $\beta := b + b'$. The group $\mathrm{TCH}^{\delta+\beta}(X \times X', \beta; m)$ consists of 0-cycles on $X \times X' \times B_\beta$. Take a closed point $P : \mathrm{Spec}(k(P)) \rightarrow X \times X_1 \times B_\beta$ as a generator and show the cycle $[P]$ is in the image of ψ . By the definition of ψ , the trace map on the additive Chow group and the norm map on the Mackey product are compatible as in the following commutative diagram:

$$\begin{array}{ccc} \mathcal{TC}\mathcal{H}^{d+b}(X_{k(P)}, b) \overset{M}{\otimes} \mathcal{CH}^{b'+d'}(X'_{k(P)}, b')(k(P)) & \xrightarrow{\psi} & \mathrm{TCH}^{\delta+\beta}((X \times X')_{k(P)}, \beta) \\ \downarrow N_{k(P)/k} & & \downarrow \mathrm{Tr}_{k(P)/k} \\ \mathcal{TC}\mathcal{H}^{d+b}(X, b) \overset{M}{\otimes} \mathcal{CH}^{b'+d'}(X', b')(k) & \xrightarrow{\psi} & \mathrm{TCH}^{\delta+\beta}(X \times X', \beta), \end{array}$$

Thus to show the surjectivity of ψ we may assume that P is a k -rational point. The point P is determined by the closed points $P_X : \mathrm{Spec} k \rightarrow X \times B_b$ and $P_{X'} : \mathrm{Spec} k \rightarrow X' \times \square^{b'}$ satisfying $\tau_*(P_X \times P_{X'}) = P$, where $\tau : (X \times B_b) \times (X' \times \square^{b'}) \rightarrow X \times X' \times B_\beta$ is the exchange of factors. This gives cycles $[P_X]$ on $\mathrm{TCH}^{d+b}(X, b; m)$ and $[P_{X'}]$ on $\mathrm{CH}^{d'+b'}(X', b')$. Therefore denoting by $p' : X \times X' \rightarrow X'$ the projection map, we have

$$\begin{aligned} \psi(\{[P_X], [P_{X'}]\}_{k/k}) \\ = (p')^*([P_{X'}]) \cap p^*([P_X]) \\ = [P]. \end{aligned}$$

□

By identifying $\mathcal{CH}^1(k, 1) \simeq \mathbb{G}_m$ and $\mathcal{TCH}^1(k, 1; m) \simeq \mathbb{W}_m$ as Mackey functors one can relate the mixed K -group defined in the last section.

Corollary 5.5. *We have surjections*

$$\begin{aligned} \mathbb{W}_m \otimes^M \overbrace{\mathbb{G}_m \otimes^M \cdots \otimes^M \mathbb{G}_m}^{q-1} \otimes^M \mathcal{CH}^d(X)(k) &\xrightarrow{\psi} \mathrm{TCH}^{d+q}(X, q; m) \\ &\xrightarrow{\phi} K(k; \mathbb{W}_m, \overbrace{\mathbb{G}_m, \dots, \mathbb{G}_m}^{q-1}, \mathcal{CH}^d(X)). \end{aligned}$$

Proof. The former surjection ψ is given by Theorem 5.4.

Definition of ϕ : The group $\mathrm{TCH}^{d+q}(X, q; m)$ is generated by classes $[P]$, where $P : \mathrm{Spec} k(P) \rightarrow X \times B_q$ is a closed point. It is determined by the closed points $P_0 : \mathrm{Spec} k(P) \rightarrow \mathbb{A}^1$, $P_i : \mathrm{Spec} k(P) \rightarrow \square^1$ ($1 \leq i \leq q-1$) and $P_X : \mathrm{Spec} k(P) \rightarrow X$. Identify the point P_0 with corresponding element in $\mathbb{G}_a(k(P))$ and do the point P_i with the element in $\mathbb{G}_m(k(P))$. The map

$$\phi : \mathrm{TCH}^{d+q}(X, q; m) \rightarrow K(k; \mathbb{W}_m, \mathbb{G}_m, \dots, \mathbb{G}_m, \mathcal{CH}_0(X))$$

is defined by sending $[P]$ to $\{[P_0], P_1, \dots, P_{q-1}, [P_X]\}_{k(P)/k}$, where $[P_0]$ is the Teichmüller lift of $P_0 \in \mathbb{G}_a(k(P))$. We have to check that the boundary of a 1-cycle $\mathrm{Tz}^{d+q}(X, q+1; m)$ maps to 0 in the Somekawa K -group. It suffices to consider an irreducible curve $C \in \mathrm{Tz}^{d+q}(X, q+1; m)$ that is an irreducible curve $C \subset X \times B_{q+1}$ in good position. The inclusion $C \hookrightarrow X \times B_{q+1}$ is defined by maps $f_0 : C \rightarrow \mathbb{A}^1 \hookrightarrow \mathbb{P}^1$, $f_i : C \rightarrow \square^1 \hookrightarrow \mathbb{P}^1$ ($1 \leq i \leq q$) and $g : C \rightarrow X$. By the very definition of $\mathrm{Tz}^{d+q}(X, q+1; m)$ at most one of f_i ($1 \leq i \leq q$) takes a value in 0 or ∞ for any closed point $P \in C$. Define

$$i(P) = \begin{cases} i, & \text{if there exists } i \text{ such that } f_i(P) \in \{0, \infty\}, \\ 1, & \text{if no such an index exists.} \end{cases}$$

Then

$$\begin{aligned} \phi \circ \partial(C) &= \sum_{P \in C_0} (-1)^{i(P)-1} v_P(f_{i(P)}) \phi(f_0(P) \times f_1(P) \times \cdots \times \widehat{f_{i(P)}(P)} \times \cdots \times f_q(P) \times g(P)) \\ &= \sum_{P \in C_0} (-1)^{i(P)-1} v_P(f_{i(P)}) \{[f_0(P)], f_1(P), \dots, \widehat{f_{i(P)}(P)}, \dots, f_q(P), s_P([\eta])\}_{k(P)/k}. \end{aligned}$$

Here we denote by $\eta : \text{Spec } k(C) \rightarrow C \xrightarrow{g} X$. Taking the normalization of C , by the projection formula (PF) we may assume that C is smooth. Now we can extend the maps f_i and g to the smooth compactification \overline{C} of C uniquely. For any closed point P in the boundary $\overline{C} \setminus C$ there exists $1 \leq j \leq q$ such that $f_j(P) = 1$ ([2], Lemma 6.6). Putting, for any $P \in C_0$,

$$j(P) = \begin{cases} 0, & \text{if } P \in \overline{C} \setminus C, \text{ or } i(P) = q, \\ i(P), & \text{otherwise.} \end{cases}$$

The above $\phi \circ \partial(C)$ can be written uniformly as

$$\sum_{P \in \overline{C}_0} \{[f_0(P)], f_1(P), \dots, \partial_P(f_{j(P)}, f_q), \dots, f_{q-1}(P), s_P([\eta])\}_{k(P)/k}.$$

Hence the Weil reciprocity law (WR) implies $\phi \circ \partial(C) = 0$. By the construction of ψ , the map ϕ is surjective. \square

From Lemma 3.5 for a perfect field k of positive characteristic we obtain the following vanishing results.

Corollary 5.6. *If k is a perfect field of positive characteristic, we have $\text{TCH}^{d+q}(X, q; m) = 0$ for $q \geq 2$.*

Further assume $X = \text{Spec}(k)$, then $\text{CH}^0(k) \simeq \mathbb{Z}$ is a unit of the product of Mackey functors and thus the surjections

$$(12) \quad \mathbb{W}_m \otimes^M \overbrace{\mathbb{G}_m \otimes^M \cdots \otimes^M \mathbb{G}_m}^{q-1}(k) \xrightarrow{\psi} \mathbb{W}_m \Omega_k^{q-1} \xrightarrow{\phi} K(k, \mathbb{W}_m, \overbrace{\mathbb{G}_m, \dots, \mathbb{G}_m}^{q-1}).$$

give (a part of) an alternate proof of Corollary 3.7.

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